Contribution à quelques problèmes de premier passage et de ruine multidimensionnels

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Contribution à quelques problèmes de premier passage et de ruine multidimensionnels ; Lien avec les réseaux de files fluides.

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Introduction

The object of the present document is to present a synthesis of the work that I started conducting after my PhD. Back then, I was not completely familiar with risk theory and was rather focusing on topics related to so-called "fluid queues" and stochastic fluid networks. Basically, a fluid queue is a one dimensional stochastic process (reflected at 0) that models the evolution of a buffer, where data arrives, and which is processed continuously. It is supposed that data is so thin that it can be represented as some fluid that arrives at some random rate in the buffer. The objective is to propose a model that matches reality as accurately as possible, and to be able to do some performance evaluation, i.e. determine quantities or stochastic measures such that distribution of the fluid queue at some fixed time or its asymptotic distribution. A proposed model, very present in my work, is a process that is modulated by an external continuous time Markov chain, which may e.g. represent the different behaviour and trends of the outside world. This model is quite flexible, and has the advantage that the fluid queue jointly to the state of the Markov chain is a Markov process. It is also possible to add a bit of complexity by adding a diffusion part to this process, that models jitter in this queue. For a state of the art on this, see [Asm03]. The methods and tools used for determining these distributions range from probability techniques (study of Markov processes, martingales, renewal theory) to numerical analysis for practical resolution of partial differential equations that appear e.g. in the Kolmogorov equations satisfied by the cdf or the Laplace transform of these distributions (see e.g. [GS07]). Extensive research is still undergone while trying to generalize the one dimensional model to reflection at boundaries, brownian motion markov modulated etc., see [Iva10, DIKM12, IP12].

A nice extension of this problem is to consider a network of such queues. Basically, what happens is that a fraction of fluid exiting one queue is directed towards one or several other queues so that, on the whole, data circulates in the network for a while before eventually departing forever. The process describing these queues is now N dimensional valued (N being the number of queues in the network). From the fact that complexity of the model increases, things naturally become more difficult. Papers dealing with such issues are less frequent (as opposed to the single server queue), see [Ram00, HMP10, PMG05].

The study of such queues lead me to study risk processes, and in particular multidimensional risk processes. Motivation is simple and comes through reinsurance : some insurance companies cannot afford to take excessive risks and need to cope with extreme events such as centennial floods, hurricanes, earthquakes etc. Hence the need to subscribe to a reinsurance contract, which will alleviate some of the incoming claims. Again, works on this domain are not as common as for the more classical one dimensional risk process. Some references are e.g. [APP08b, APP08a, BLMV10, Bia10, GBC12, HJ13, Ram12], though the list is not exhaustive.

Organization of the manuscript. The first Chapter is partly devoted to study of some particular fluid queues in Sections 1.2 and 1.3. Concerning this topic, they cover articles [RS04, Rab06b, Rab06a]. These papers were published after my thesis, but contain some material from my thesis. In fact, one of the motivations for this chapter is to establish a link with ruin theory that may have first been established by Asmussen and Schock Petersen [ASP88], and to try to broaden this correspondance to the multidimensional setting : see in particular Theorems 5 and 6 in Section 1.3.1. In that respect, results from [RS04, Rab06b, Rab06a] are seen a bit differently, as it is attempted to see how those results are read from a risk theory point of view. Let us emphasize out that, at this point, this is just an attempt, as it is a link between stochastic networks and rather simple (and not so standard) ruin problems. However, this hopefully should lead in the future to interesting perspectives. Likewise, Section 1.4 (which presents the first part of [Rab09]) starts by considering a one dimensional risk process with interest rate, then shifts to see how some exit times of a corresponding N dimensional process with interest rate and claims occurring along the same process (but with different sizes) can be obtained as a (not straightforward) consequence.

The second chapter deals again with multidimensional risk theory, but focuses on the case where claims arrive according to one source and are split among one or several (sub)companies. This part covers the second part of [Rab09] as well as [BCR11] and [Rab12]. The reason why a chapter is dedicated to such a scenario is that it is the most favorable case leading to explicit formulas thanks to a geometric interpretation of the problem, as seen in Sections 2.1 and 2.2. In these sections, particular attention is given to the two dimensional case for technical reasons, one attempts to reduce the 2 dimensional problem to a series of easier problems, mainly by pointing out absorbing sets for the bivariate risk process. When all else fails, one turns to asymptotics in Section 2.3, where exact results seem anyway very difficult to derive because the model includes a fractional brownian motion.

Finally, the last chapter addresses some other works, and covers [GR07, RCLT13, ALR09, PR13]. Section 3.1 sees an application of theory of admission control to a fluid queue problem, and sees (via the duality result) how this translates to the risk theory setting. It basically discusses the problem of how to plan acceptance/rejection of incoming packets of data (fluid) to the queue so as not to asymptotically accept less than a given proportion p of packets. In Section 3.2 a question that actually came from an N dimensional ruin problem is raised, which how to know the join distribution of the ruin time and the amount of money that was spent up to that time ; tools for solving this problem are standard and use optional stopping theorem, integro differential equations, and embedding. Section 3.3 sees how to try to determine when the ruin probability can be written as a so called Erlang expansion, in the case when there is a diffusion coefficient which is quadratic with respect to the capital level. Finally, Section 3.4 sees a problem closely related to risk theory, which deals with how to obtain closed expressions on the first and last passage of a general spectrally Lévy process above a fixed level. This has some application in reliability, where this process can be seen as the degradation level of a certain item.

A final word about the presentation of the document : all results published in my articles and mentioned here are presented with only hints of the proofs (not full ones), trying to emphasise their general backbones and highlight where the most delicate points stand out. Only Theorems 5 and 6 feature full (short) proofs, which rely heavily on some results from [Rab06b]. Those Theorems are completely new and are, from my point of view and along with the discussions on multi dimensional risk theory at the end of Sections 1.3.2 and 1.3.3, a starting point for future research. Furthermore, I deliberately did not mention an ongoing work (even though it is very enjoyable) on a topic related to branching random walks and which originally stemmed from a common project initiated with other teams in a Computer Science and Biology/Ecology departments here at the Université of Franche Comté. A remark on this aspect, as well as some ideas for future research, is given at the end of the document in the "Perspectives and Ouvertures" chapter.

Key words : Applied probability and stochastic modelling, First passage problems in dimension 1 and $N \geq 2$ and applications to risk theory, Fluid queues networks.

List of published or accepted articles in journals.

The following list is in chronological order. Most of them are mentioned again in the Reference (Bibliographie) part of the document.

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jump process, in C.Bérenguer, A.Grall and C.G.Soares, editors, *Risk, Reliability and Societal Safety, Proceedings of the European Safety and Reliability Conference 2011, Troyes, France, 18–22 September 2011 (ESREL 2011)*, Taylor and Francis.

Other documents

[15] Contejean E., Marché C. and Rabehasaina L. Rewrite systems for natural, integral, and rational arithmetic, *Lecture Notes in Comput. Sci.*, **1232** (1997), pp.98–112.

[16] Avram F., Biard R., Dutang C., Loisel S. and Rabehasaina L. A survey of some recent results on Risk Theory, to appear in *ESAIM proceedings* (2013).

Some remarks on those two last references : [15] was published as a consequence of a report I wrote (under the supervision of Claude Marché) during an internship at the Laboratoire de Recherche en Informatique at the Université Orsay Paris 11 in July 1996, during my undergraduate studies. [16] is an account of the talks that were given during the "Journée Modélisation Aléatoire et Statistique" that took place in Clermont Ferrand in August 2012.

Chapitre 1

From Fluid queues to Risk theory : Duality revisited

1.1 Introduction

We present two tools that essentially provide a link between queueing and risk theory. The first one finds mainly its origins in a paper by Asmussen and Schock Petersen [ASP88] (and was already mentioned in [Sea72] for an M/G/1 queue), and found many applications and interpretations in subsequent papers [SR00], [BLP11]. The second is rather standard and transforms a discontinuous one into a continuous one, see [AAU02, BBdSS+05].

1.1.1 Duality

Let $\{R_t,\ t \geq 0\}$ be a risk process satisfying the following dynamics :

$$\left\{\begin{array}{rcl} dR_t & = & p(R_t)dt - dS_t \\ R_0 & = & u \end{array}\right. \tag{1.1}$$

For some positive continuous function $p(.)$ (the premium rate), initial reserve $u \geq 0$, and compound Poisson process $\{S_t,\ t \geq 0\}$ (the aggregate claim amount). We then define the *dual* queueing process $\{Q(t),\ t \geq 0\}$ related to $\{R_t,\ t \geq 0\}$, which verifies

$$\left\{\begin{array}{rcl} dQ(t) & = & -p(Q(t))dt + dS_t + dL_t \\ L_t & = & \displaystyle\int_0^t 1_{\{Q(s)=0\}} dL_s \end{array}\right. \tag{1.2}$$

where $\{L_t,\ t \geq 0\}$, often called the *compensator*, is a non decreasing process that prevents $Q(t)$ from becoming negative, so that the solution to (1.2) is a couple $\{(Q(t), L_t),\ t \geq 0\}$. Let us define the *ruin time* of risk process R_t by

$$\tau := \inf\{t \geq 0 |\ R_t < 0\}.$$

Asmussen and Schock Petersen prove in a short note the following theorem that links distribution of τ and $\{Q(t),\ t \geq 0\}$.

Theorem 1 ([ASP88]). *Let $T \in (0, +\infty]$ and $\psi(u, T) := \mathbb{P}(\tau \leq T | R_0 = u)$ the finite horizon ruin probability. Then*

$$\psi(u, T) = \mathbb{P}(Q(T) > u). \tag{1.3}$$

where $Q(\infty)$ is understood as the limiting distribution of $Q(t)$ as $t \to +\infty$.

This result can be extended to more complicated models, e.g. a risk process with a diffusion coefficient say $\sigma(R_t)dB_t$ for some brownian motion $\{B_t, \ t \geq 0\}$. However, one has to be careful on defining the corresponding queueing dual process while reversing time, as the Ito integral $\sigma(R_t)dB_t$ is transformed into a different (non anticipative) integral for $Q(t)$, see Proposition 4.3 of [SR00].

1.1.2 Embedding

A particular case is when claims are Phase type distributed. Let us recall that a random variable variable X is Phase type distributed with representation (γ, G, t) if it is the absorbing time of a continuous time Markov chain of state space $\{1, ..., n+1\}$, with intensity matrix defined blockwise by

$$\begin{pmatrix} G & t \\ 0 & 0 \end{pmatrix},$$

where G is a subintensity $n \times n$ matrix, γ and t are vector columns of size n with γ being a probability vector and $t = -G1$ (1 being the column vector of size n with all entries equal to 1). We will say in short $X \sim PH(\gamma, G, t)$, see Section 1 of Chapter IX in [AA10]. It is then possible to define a new continuous risk process that is such that :

- it evolves the same way as the original risk process in between claim occurences,
- downward jumps due to claims are replaced by oblique lines with slope $-1/a$ for some $a > 0$.

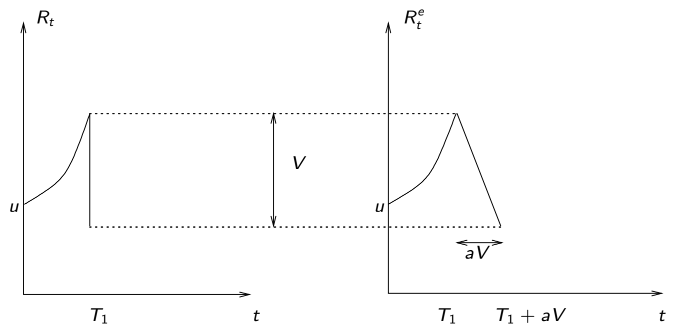

FIGURE 1.1 – Original risk process and corresponding continuous process R_t

Letting $\{R_t^e, \ t \geq 0\}$ this continuous, embedded process, one has that it verifies $R_0^e = u$ and

$$dR_t^e = p(J(t), R_t^e)dt \tag{1.4}$$

where $\{J(t), \ t \geq 0\}$ is a Markov chain defined on an appropriate state space, and function $p(.,.)$ is defined such that $p(J(t), R_t^e) = -1/a$ whenever $J(t)$ is in a state corresponding to occurrence of a claim, and $p(J(t), R_t^e) = p(R_t^e)$ otherwise, see Figure 1.1. This construction is very standard, and is mainly motivated by the fact that the embedded process is a Markov modulated process, which is a favorite topic in the Matrix Analytic Method community, see for instance [AAU02, BBdSS+05] as well as Chapter VII of [AA10]. Value of $a > 0$ is, in the latter papers, equal to 1. Introducing this extra parameter will enable us to get additional information on quantities other than the ruin time of process $\{R_t,\ t \geq 0\}$. Let us note that, by construction, ruin time of the embedded process similarly defined as

$$\tau^e := \inf\{t \geq 0 |\ R^e_t < 0\}$$

is finite if and only ruin time of the original process is finite, which translates as, with the definition given in Theorem 1, $\mathbb{P}(\tau < +\infty |\ R_0 = u) = \psi(u, +\infty)$. Finally, the analog counterpart of the dual queueing process (1.2) can also be similarly constructed and is commonly called a Markov modulated fluid queue, see e.g. Chapter 6 of [Neu81].

In the sequel, one will often use the same notation for the risk process or queueing process and its embedded version, when no confusion is possible.

1.2 Fluid queue approach in dimension 1

In [RS04] we are interested in Markov modulated, embedded and diffusive version of (1.1) and (1.2). The queueing process $\{Q(t),\ t \geq 0\}$ verifies the following linear stochastic differential equation

$$dQ(t) = \lambda(X(t))dt - \mu(X(t))Q(t)dt + \sigma(X(t))Q(t)dB_t, \tag{1.5}$$

where $\{X(t),\ \in \mathbb{R}\}$ and $\{B_t,\ t \in \mathbb{R}\}$ are respectively two sided (out of convenience) irreducible finite stationary Markov chain and brownian motion. Note that, since $X(t)$ has finite state space, drift and diffusion coefficients in (1.5) are uniformly Lipschitz with respect to $Q(t)$, so that (1.5) admits a unique solution for a fixed initial condition $Q(0)$. State space of the Markov chain is denoted by $S = \{1, ..., N\}$, its generating matrix is given by $Q = (q_{ij})_{i,j \in S^2}$, and its (stationary) distribution is denoted $\pi = (\pi_i)_{i \in S}$. $\lambda(.)$, $\mu(.)$, $\sigma(.)$ are non negative functions. Due to the form of the stochastic differential equation satisfied by $Q(t)$, one has that $Q(t) \geq 0$ for all $t \geq 0$, so that compensator $\{L_t,\ t \geq 0\}$ in (1.2) is in fact equal to zero. In that case the corresponding dual embedded risk process verifies

$$dR_t \quad = \quad \rho(J(t))R_t dt + s(J(t))R_t dB_t - \nu(I(t))dt \tag{1.6}$$

where $\{J(t), \geq 0\}$ is a finite Markov chain which has the same distribution as a reversed version of $\{X(t),\ \in \mathbb{R}\}$, $\lambda(.) = \nu(.)$, $\mu(.) = \rho(.) - s(.)^2$, $s(.) = \sigma(.)$. In that case, one sees, comparing (1.6) to (1.4), that R_t is a particular risk process with no premium, and only subject to risky investment and claims which are Phase type distributed. Although this model is not very common from an insurance point of view, the equivalent of (1.3) nonetheless holds and one has, letting

$$W \stackrel{\mathcal{D}}{:=} \lim_{t \to +\infty} Q(t), \quad \tau := \inf\{t \geq 0 |\ R_t < 0\},$$

respectively the limiting distribution of the queue content and the ruin time, the duality equality

$$\mathbb{P}(\tau < +\infty |\ R_0 = u) = \mathbb{P}(W > u). \tag{1.7}$$

The aim of [RS04] is to give a closed expression of the first two moments of W. From a queueing point of view, getting information on the expectation and variance of the steady limiting state of the queue level is important. From a risk theory point of view, second moment of W can lead to an upper bound of the probability of eventual ruin, thanks to Tchebychev's inequality and (1.7). Due to linearity of the stochastic differential equation (1.5), it turns out that W has a nice expression which is given in the following result :

Theorem 2. *Let us set for all $t \in \mathbb{R}$*

$$W(t) = \int_{-\infty}^{t} \exp\left(-\int_{s}^{t}[\mu(X(v)) + \sigma(X(v))^2/2]dv + \int_{s}^{t}\sigma(X(v))dB_v\right)\lambda(X(s))ds. \quad (1.8)$$

Then

1. *$W(t)$ is finite for all t.*
2. *$\{W(t),\ t \in \mathbb{R}\}$ is a stationary process solving (1.5).*
3. *$Q(t)$ converges in distribution to $W(0) := W$, independently of the initial condition $Q(0)$.*

The main steps for proving Theorem (2) are the following. We first note that solution to (1.5) is standard and is given by

$$Q(t) = Q(0)\exp\left(-\int_{0}^{t}[\mu(X(s)) + \sigma(X(s))^2/2]ds + \int_{0}^{t}\sigma(X(s))dB_s\right), \quad t \geq 0.$$

Having this expression in mind, one then introduces family of processes $\{Q_u^y(t),\ t \geq u\}$, $u \in \mathbb{R}$, $y \geq 0$, that satisfy (1.5) for all $t \geq u$ with $Q_u^y(u) = y$, so that, similarly,

$$\begin{aligned} Q_u^y(t) &= y\exp\left(-\int_{u}^{t}[\mu(X(s)) + \sigma(X(s))^2/2]ds + \int_{u}^{t}\sigma(X(s))dB_s\right) \\ &+ \int_{u}^{t}\exp\left(-\int_{s}^{t}[\mu(X(v)) + \sigma(X(v))^2/2]dv + \int_{s}^{t}\sigma(X(v))dB_v\right)\lambda(X(s))ds. \quad (1.9) \end{aligned}$$

The main point of proof Theorem (2) is to be able to justify letting $u \to -\infty$ in (1.9).

We let $Q^* = (q_{ij}^*)_{i,j\in S}$ be the generating matrix associated to the reversed Markov process $\{X^*(t) := X((-t)^+),\ t \geq 0\}$. The relation between Q and Q^* is given by $q_{ij}^* = \pi_j q_{j,i}/\pi_i$. Likewise, we let $\{B_t^* := B_{-t},\ t \geq 0\}$ the reversed version of $\{B_t,\ t \geq 0\}$ (both processes have same distribution). The first moment is given by the following result.

Theorem 3. *For all $h : S \longrightarrow \mathbb{R}$,*

$$\mathbb{E}(W\,h(X(0))) = \pi H\,(D_\mu - Q^*)^{-1}\,\Lambda 1 \quad (1.10)$$

where $H = diag\,(h(1), ..., h(N))$, $D_\mu = diag\,(\mu(1), ..., \mu(N))$, $\Lambda = diag\,(\lambda(1), ..., \lambda(N))$ and $1 = (1, ..., 1)'$. In particular, if $h(i) = 1$ for all $i \in S$ we get the expression of the first moment of W :

$$\mathbb{E}(W) = \pi\,(D_\mu - Q^*)^{-1}\,\Lambda 1 \quad (1.11)$$

Expression (1.11) is reminiscent of Expression (4) in [KS02], however with an original model with Brownian component, which does not show in (1.11). The second moment exists and is given under the following technical assumption that function $\sigma(.)$ is not too large :

Assumption 1. $\forall i \in S, \quad \mu(i) \geq 4\sigma(i)^2$ *and* $\exists i \in S, \quad \mu(i) > 4\sigma(i)^2$.

Theorem 4. *Let us denote $D_{2\mu-\sigma^2} := diag\,(2\mu(1) - \sigma(1)^2, ..., 2\mu(N) - \sigma(N)^2)$, $D_{\mu-2\sigma^2} := diag\,(\mu(1) - 2\sigma(1)^2, ..., \mu(N) - 2\sigma(N)^2)$. Then, under Assumption 1, $\mathbb{E}(W^2)$ is finite and has the* *following expression*

$$\begin{aligned}\mathbb{E}(W^2) &= 2\pi(D_{2\mu-\sigma^2} - Q^*)^{-1}(Q^* - D_{\mu-4\sigma^2})^{-1}\Lambda^2 1 \\ &- 2\pi(Q^* - D_{\mu-4\sigma^2})^{-1}\Lambda(D_\mu - Q^*)^{-1}\Lambda 1. \end{aligned} \tag{1.12}$$

Let us skip proof of Theorem 3 and present a sketch of proof of Theorem 4. Assumption 1, at the time of writing the article, was a sufficient condition ensuring finiteness of $\mathbb{E}(W^2)$. No further thought were given ever since it was published, however it may be on closer look of the proof that this condition is in fact necessary.

Having proved that second moment exists, we introduce the family of processes $\{Z_t^u,\ t \geq u\}$, $u \in \mathbb{R}$, defined by the following linear equation

$$\begin{cases} dZ_t^u &= 2\lambda(X(t))W(t)dt + [-2\mu(X(t)) + \sigma(X(t))^2]Z_t^u dt + 2\sigma(X(t))Z_t^u dB_t \\ Z_u^u &= 0. \end{cases} \tag{1.13}$$

One observes resemblance between (1.13) and (1.5). The main difference being, that (1.13) features the stationary process $\{W(t),\ t \in \mathbb{R}\}$, of which not a lot is known at this stage. In view of similarity between $\{Z_t^u,\ t \geq u\}$ and $\{Q_u^y(t),\ t \geq u\}$, and of expression (1.8), the following lemma is not surprising :

Lemma 1. *Z_t^0 converges in distribution as $t \to -\infty$ towards*

$$\int_{-\infty}^0 \exp\left(-\int_s^0 [2\mu(X(v)) + \sigma(X(v))^2]dt + \int_s^0 2\sigma(X(v))dB_v\right)\ 2\lambda(X(s))W(s)ds. \tag{1.14}$$

Besides, this random variable is equal to W^2 in distribution.

The rest of the proof consists in computing $\mathbb{E}(W^2)$ by taking the expectation of (1.14), which itself features process $\{W(t),\ t \in \mathbb{R}\}$. We first make the change of variable $t := -t$ in (1.14) and obtain that W^2 has same distribution as

$$\int_0^\infty \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2]dt + \int_0^s 2\sigma(X^*(v))dB_v^*\right)\ 2\lambda(X^*(s))W^*(s)ds. \tag{1.15}$$

where $W^*(s) := W(-s)$, which in turn verifies from (1.8)

$$W^*(s) = \int_s^\infty \exp\left(-\int_s^r [\mu(X^*(h)) + \sigma(X^*(h))^2/2]dh + \int_s^r \sigma(X^*(h))dB_h^*\right)\lambda(X^*(r))dr.$$

We also let

$$M^*(s) := \exp\left(-\int_0^s [2\mu(X^*(v)) + \sigma(X^*(v))^2]dv + \int_0^s 2\sigma(X^*(v))dB_v^*\right),$$

so that (1.15) reads $\displaystyle\int_0^\infty M^*(s)\ 2\lambda(X^*(s))W^*(s)ds.$

Generator $\mathcal{A}'$ of Markov process $\{(W^*(t), M^*(t), X^*(t)),\ t \geq 0\}$ is given by

$$\begin{aligned}\mathcal{A}'g(x_w, x_m, i) &= [(\mu(i) + \sigma(i)^2)x_w - \lambda(i)]\partial_{x_w} g(x_w, x_m, i)\\ &- [2\mu(i) - \sigma(i)^2]x_m \partial_{x_m} g(x_w, x_m, i)\\ &+ \frac{1}{2}\sigma(i)^2 x_w 2\partial^2_{x_w} g(x_w, x_m, i) + 2\sigma(i)^2 x_m 2\partial^2_{x_m} g(x_w, x_m, i)\\ &+ 2\sigma(i)^2 x_m x_w \partial_{x_m}\partial_{x_w} g(x_w, x_m, i) + \sum_{j\in S} q^*_{ij} g(x_w, x_m, j)\end{aligned}$$

for $x_w \geq 0, x_m \geq 0, i \in S$ and smooth enough $g : \mathbb{R}\times\mathbb{R}\times S \longrightarrow \mathbb{R}$. The rest of the proof consists in finding an appropriate function $g(.,.,.)$ solving

$$\mathcal{A}'g(x_w, x_m, i) = 2x_m\lambda(i)x_w, \quad \forall x_w \geq 0, x_m \geq 0, i \in S,$$

so that, after verifying that Dynkin's formula is valid, one has

$$\begin{aligned}\mathbb{E}\left(\int_0^t M^*(s).\ 2\lambda(X^*(s))W^*(s)ds\right) &= \mathbb{E}\left(\int_0^t \mathcal{A}'g(W^*(s), M^*(s), X^*(s))\right)\\ &= \mathbb{E}(g(W^*(t), M^*(t), X^*(t))) - \mathbb{E}(g(W^*(0), M^*(0), X^*(0))) \quad (1.16)\end{aligned}$$

The righthandside of (1.16) converges to $\mathbb{E}(W^2)$ as $t \to +\infty$. As to the righthandside, one proves that term $\mathbb{E}(g(W^*(t), M^*(t), X^*(t)))$ converges to zero and that $\mathbb{E}(g(W^*(0), M^*(0), X^*(0)))$ can be computed and is actually equal to Expression (1.12) thanks to Theorem 1.10 applied to a similar queueing process as (1.5) with different parameters.

1.3 Stability of stochastic networks and ruin probability in dimension $N \geq 2$

We wish in this section to see how notion of duality exposed in Section 1.1.1 may be adapted to a multidimensional setting. Articles presented in this section are [Rab06b] and [Rab06a]. At the time when they were published, focus was primarily on stochastic networks, i.e. a queueing context. In the subsequent sub section, we will present main results of those papers, but we will also show how they can be seen from a risk theory point of view.

The model considered in [Rab06b] and [Rab06a] is that of an N dimensional network of contents $Q(t) = (Q^1(t), ..., Q^N(t))'$ that satisfies the analog of (1.2) :

$$\begin{cases} dQ(t) = b(X(t), Q(t))dt + \Sigma(X(t), Q(t))dB_t + (I - P')dL(t) & \text{for all } t \geq 0\\ Q(t) \geq 0 \quad \text{for all } t \geq 0 & \\ L^i(t) = \int_0^t 1_{\{Q^i(s)=0\}} dL^i(s) \quad \text{for all } t \geq 0, & i = 1, ..., N\end{cases} \quad (1.17)$$

where

- The process $\{X(t),\ t \in \mathbb{R}\}$ is ergodic with state space $\mathbf{X}$.
- $\{B_t = (B^1_t, ..., B^N_t)',\ t \in \mathbb{R}\}$ is an N dimensional Brownian motion with independent entries, independent from $\{X(t),\ t \in \mathbb{R}\}$.
- For all $x \in \mathbf{X}$ and $y = (y^1, ..., y^N)'$, $\Sigma(x, y) :=$ diag $(\sigma^1(x, y^1), ..., \sigma^N(x, y^N))$ is a diagonal matrix, and $b(x, y) = (b^1(x, y^1), ..., b^N(x, y^N))'$ is a column vector.

– b^i, and σ^i, $i = 1, ..., N$, are bounded and Lipschitz functions with respect to the last N variables, i.e. there exists $C > 0$ such that for all $(x, y^1, ..., y^N, z^1, ..., z^N) \in \mathbf{X} \times [0, +\infty)^{2N}$

$$\sum_{i=1}^{N} |b^i(x, y^1, ..., y^N) - b^i(x, z^1, ..., z^N)| \leq C \left(\sum_{i=1}^{N} |y^i - z^i| \right)$$
$$\sum_{i=1}^{N} |\sigma^i(x, y^i) - \sigma^i(x, z^i)| \leq C \left(\sum_{i=1}^{N} |y^i - z^i| \right)$$

– $P = (p_{ij})_{i,j=1,...,N}$ is a matrix with $p_{ii} = 0$, $p_{ij} \in [0, 1]$ and $\sum_{j \neq i} p_{ij} \leq 1$ and that verifies the property

$$P^n \to 0, \quad n \to \infty, \tag{1.18}$$

so that $(I - P)^{-1} = \sum_{n=0}^{\infty} P^n$ exists and is non-negative. We say that P is an M-matrix, as defined in [CY01].

P will be referred to as the *routing matrix*. Practically speaking, p_{ij} corresponds in a queueing network to the fraction of fluid (data) issued from queue i that is rerouted to queue j. Its interpretation will be especially made clear in Theorem 9 in the following subsection 1.3.2.

N dimensional process $\{L(t) = (L^1(t), ...L^N(t))', \ t \geq 0\}$ is, as in dimension 1 in (1.2), componentwise non decreasing. Note that at first sight, it is not obvious that (1.17) admits a strong solution $\{(Q(t), L(t)), \ t \geq 0\}$. Yamada [Yam95] proved that, under the suitable Lipschitz conditions, and when P is indeed an M-matrix, then indeed a solution to (1.17) exists and is unique when initial condition $Q(0)$ is fixed.

$\{X(t), \ t \in \mathbb{R}\}$ is commonly referred to the *environment* in the queueing theory literature. Although, as one will see, it does not add a lot to the technicality to subsequent proofs, it plays an important role from the modelling point of view, since it represents randomness due to exogenous factors.

We will make one of the following assumption

Assumption 2. – $N = 2$
– *$b^1(x, y^1, .)$ and $b^2(x, ., y^2)$ are non decreasing, for all $x \in \mathbf{X}$, $y^1 \geq 0$, $y^2 \geq 0$.*

Assumption 3. – *$\Sigma(., .) \equiv 0$,*
– *$b^i(x, y^1, ..., y^N) \geq 0$ for all $x \in \mathbf{X}$, $y^1 \geq 0$,..., $y^N \geq 0$,*
– *for each $i = 1, ..., N$ and each $j \neq i$, $b^i(x, y^1, ..., y^N)$ is non decreasing in y^j.*

In the following we will write $v \geq u$ for two vectors v and u to express that each component of v is larger than the corresponding component of u.

1.3.1 From N dimensional fluid network to N dimensional ruin problems.

We now motivate a potential link with multidimensional ruin theory by presenting the dual embedded risk process in the particular case when Assumption 2 holds, and the external environment $\{X(t), \ t \in \mathbb{R}\}$ is a finite stationary ergodic Markov chain. This risk process $\{R_t = (R_t^1, R_t^2)', \ t \geq 0\}$ satisfies, similarly to (1.1) and with the embedding construction explained in Section 1.1.2,

$$\begin{cases} dR_t &= -b(X^*(t), R_t)dt = -b(X^*(t), R_t^1, R_t^2)dt \\ R_0 &= (u^1, u^2)' \end{cases} \tag{1.19}$$

where $X^*(t) = X((-t)^+)$ is the reversed Markov chain (as defined shortly after Theorem 2). Remember that $\{R_t,\ t \geq 0\}$ is the embedded process, so that drift $b(X^*(t), R_t)$ accounts for evolution of the process in between claims and during claims. The simplest example (which will be detailed in the two next sub sections as applications) is the case where $-b(X^*(t), R_t) = -1$ when $X^*(t)$ is in a state corresponding to occurrence of a claim, and $-b(X^*(t), R_t)$ is the drift of the risk process while $X^*(t)$ is in a state corresponding to evolution in between claims. We let

$$\tau^i := \inf\{t \geq 0 |\ R_t^i < 0\}, \quad i = 1, 2.$$

Note that no matrix P is present in (1.19) (only the drift function as well as the environment is specified), so that $\{R_t,\ t \geq 0\}$ is defined as the dual process of a process $\{Q(t),\ t \geq 0\}$ corresponding to any reflecting matrix P. This actually gives a bit of flexibility, as different results (which rely on upcoming comparison Theorem 7) are given according to whether this matrix is zero or not. We present two types of results corresponding to these two cases.

Case $P = 0$. Here we have a bit of freedom and can describe what happens after one branch hits zero, i.e. after $\min(\tau^1, \tau^2)$ (corresponding to hitting frontier of the first quadrant). We define ruin time ν of process $\{R_t = (R_t^1, R_t^2),\ t \geq 0\}$ as hitting time of point $(0, 0)'$, as follows. If say R_t^1 hits 0 first (i.e. $\tau^1 \leq \tau^2$) then it is killed and second process evolves according to the equation

$$dR_t^2 = -b^2(X^*(t), R_t^1, R_t^2)dt = -b^2(X^*(t), 0, R_t^2)dt, \quad t \geq \tau^1 = \min(\tau^1, \tau^2), \tag{1.20}$$

then hits potentially zero at some random time ν. See Figure 1.2 for an illustration. If R_t^2 hits 0

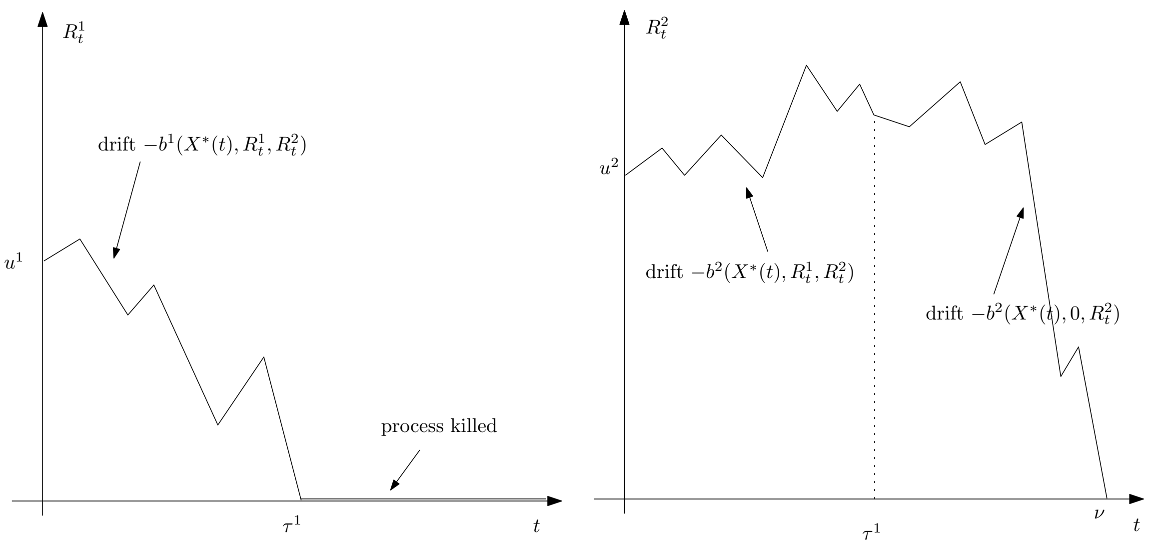

FIGURE 1.2 – Sample paths of $\{R_t^1,\ t \geq 0\}$ and $\{R_t^2,\ t \geq 0\}$ and ruin times.

first then ν is defined the same way by swapping roles of R_t^1 and R_t^2. Then a partial analog of Theorem 1 holds :

Theorem 5. *Let $P = 0$ and $T \in (0, +\infty)$ and $\psi(u, T) := \mathbb{P}(\nu \leq T \mid R_0 = u)$ the finite horizon ruin probability. Then*

$$\psi(u, T) \leq \mathbb{P}(Q^1(T) \geq u^1, Q^2(T) \geq u^2) \leq \mathbb{P}(\min(\tau^1, \tau^2) \leq T \mid R_0 = u) \tag{1.21}$$

(1.21) also holds with $T = +\infty$, where $Q(\infty) = (Q^1(\infty), Q^2(\infty))$ is understood as the limiting distribution of $Q(t)$ as $t \to +\infty$.

Proof. We use comparison Theorem 7 (mentioned thereafter) in its simplest form when $P = 0$. Since the analysis will be on sample paths we suppose w.l.o.g. that on $[\nu \leq T]$ we have $\tau^1 \leq \tau^2$. The starting point is approximately the same as in Theorem 2 : we define for all $u \in \mathbb{R}$ the 2 dimensional process $\{Q_t^u = (Q_t^{1,u}, Q_t^{2,u})', \ t \geq u\}$ that verifies $Q_u^u = (0,0)'$ and verifies (1.17) for $t \geq u$ (replacing 0 by u in that equation), along with corresponding compensator $\{L_t^u = (L_t^{1,u}, L_t^{2,u})', \ t \geq u\}$ satisfying $L_u^u = 0$. Let us also define $\{Y_t, \ t \in [-\nu, 0]\}$ by $Y_t := R_{-t}$. We note that

- solution $\{(Z_t, L_t), \ t \in [-\nu, -\tau_1]\}$ of $dZ_t = b^2(X(t), 0, Z_t) + dL_t$ with $Z_{-\nu} = 0$ is $Z_t = Y_t^2$ and $L_t = 0$ (since $Y_t^2 > 0$ on $(-\nu, -\tau^1]$),
- $Q_{-\tau^1}^{2,-\nu} \geq Y_{-\tau^1}^2$, as indeed the fact that $b^2(x, ., y^2)$ is increasing implies that drift $b^2(X(t), Q_t^{2,-\nu}, .)$ of $Q_t^{2,-\nu}$ on $[-\nu, -\tau^1]$ is larger than drift $b^2(X(t), 0, .)$ of Y_t^2 on the same interval, and since $Y_{-\nu}^2 = Q_{-\nu}^{2,-\nu} = 0$. [1]

Since in addition $Q_{-\tau_1}^{1,-\nu} \geq 0 = Y_{-\tau^1}^1$, this entails by Theorem 7 that $Q_t^{-\nu} \geq Y_t$ componentwise on $t \in [-\tau^1, 0]$. In particular, one has $Q_0^{-\nu} \geq Y_0 = (u_1, u_2)'$. Since again by comparison Theorem 7 one has $Q_t^{-T} \geq Q_t^{-\nu}$ on $t \in [-\nu, 0]$, one finally obtains on event $[\nu \leq T, \ \tau^1 \leq \tau^2]$

$$Q_0^{-T} \geq Q_0^{-\nu} \geq (u_1, u_2)',$$

which is also valid on $[\nu \leq T, \ \tau^1 \geq \tau^2]$ by switching roles of $Q_t^{1,-T}$ and $Q_t^{2,-T}$, so holds on $[\nu \leq T]$.

Now let us suppose that $Q_0^{-T} \geq (u^1, u^2)'$ and prove that $\min(\tau^1, \tau^2) \leq T$. We actually prove the stronger fact that $\min(\tau^1, \tau^2) > T \Rightarrow Q_0^{-T} < (u^1, u^2)'$. So suppose that $\min(\tau^1, \tau^2) > T \iff R_t > 0$ for all $t \in [0, T]$. We set, as above, $Y_t := R_{-t}$ on $t \in [-T, 0]$, so that $Y_t > 0$ for all $t \in [-T, 0]$. We remark that $\{Y_t, \ t \in [-T, 0]\}$ is solution to (1.17) with corresponding compensator $L_t = L_t^Y \equiv 0$ (remember that Y_t is positive, $t \in [-T, 0]$, so that compensation is not necessary), and with $Y_{-T} = R_T > 0$. Since $\{Q_t^{-T}, \ t \in [-T, 0]\}$ also verifies (1.17) and $Q_{-T}^{-T} = 0 < Y_{-T}$, we thus obtain by Theorem 7 that, at time 0, $Q_0^{-T} < Y_0 = R_0 = (u^1, u^2)'$.

We then have, all in all,

$$[\nu \leq T] \subset [Q_0^{-T} \geq (u^1, u^2)'] \subset [\min(\tau^1, \tau^2) \leq T].$$

Since $\{X(t), \ t \in \mathbb{R}\}$ is stationary, Q_0^{-T} is equal in distribution to Q_T^0. (1.21) follows. □

Case $P \neq 0$. We have the following result.

Theorem 6. *Let P be an M-matrix and $T \in (0, +\infty]$. One has the following bound for the probability of hitting the frontier of the first quadrant*

$$\mathbb{P}(\min(\tau^1, \tau^2) > T) \leq \mathbb{P}(Q^1(T) < u^1, Q^2(T) < u^2) \tag{1.22}$$

Proof. Proof of this result lies on the remark that Proof of inclusion $[\min(\tau^1, \tau^2) > T] \subset [Q_0^{-T} < (u^1, u^2)']$ in Theorem 5 is valid for any reflection matrix P for process $\{Q(t), \ t \geq 0\}$ (not just $P = 0$). We conclude again by the fact that $Q_0^{-T} \overset{\mathcal{D}}{=} Q_T^0$ to obtain (1.22). □

1. Note that this is at this point that we need to assume that reflection matrix P is zero. We indeed use here the famous standard comparison result that states that if two one dimensional, reflected at 0, processes Y_t^1 and Y_t^2 satisfy $dY_t^i = b^i(t, Y_t^i)dt + dL_t^i$ with $Y_0^1 \leq Y_0^2$ and their drifts verify $b^1(t, .) \leq b^2(t, .)$ for all $t \geq 0$ then $Y_t^1 \leq Y_t^2$ for all $t \geq 0$, see [EKCM78].

We conclude by remarking that P is general in Theorem 6, but that the consequence is that (1.22) is less accurate than (1.21). Both results are however compatible as, indeed, $\mathbb{P}(\min(\tau^1, \tau^2) \leq T) \geq \mathbb{P}(\nu \leq T)$.

1.3.2 Existence of an asymptotic distribution, and an application to risk theory.

We focus on model (1.17), with one of the Assumptions 2 or 3 holding. The objective of [Rab06b] is to prove existence of a limiting distribution for $Q(t) = (Q^1(t), ..., Q^N(t))'$ as $t \to +\infty$. From a queueing point of view, such a result is important as this helps to know if congestion is almost sure or not in the long run. All results presented in the next two results focus on $N = 2$. A risk theory application will be given at the end of this section.

Let us define $\tilde{b}(x, y^1, y^2) = (\tilde{b}^1(.,.,.), \tilde{b}^2(.,.,.))'$, $i = 1, 2$, by

$$\tilde{b}(x, y^1, y^2) := (I - P')^{-1} b(x, y^1, y^2).$$

The main results of [Rab06b] are the two following :

Theorem 7. *Let us suppose that Assumption 2 holds. Let* $(Y_t, K_t) = ((Y_t^1, Y_t^2), (K_t^1, K_t^2))$ *and* $(Z_t, L_t) = ((Z_t^1, Z_t^2), (L_t^1, L_t^2))$ *be two solutions to (1.17) satisfying* $Y_0 \geq Z_0$. *Then* $Y_t \geq Z_t$ *and* $L_{t+h} - L_t \geq K_{t+h} - K_t$ *for all* $t, h \geq 0$.

Theorem 8. *Let us suppose that Assumption 2 holds and*

$$\mathbb{E}\left(\limsup_{y^1\to\infty} \sup_{y^2\geq 0} \tilde{b}^1(X(0), y^1, y^2)\right) < 0 \quad \text{and} \quad \mathbb{E}\left(\limsup_{y^2\to\infty} \sup_{y^1\geq 0} \tilde{b}^2(X(0), y^1, y^2)\right) < 0. \tag{1.23}$$

Then there exists a stationary process solution to (1.17). More precisely, there exists an a.s. finite nonnegative stationary process $\{W(t) = (W^1(t), W^2(t))', \ t \in \mathbb{R}\}$, *and a couple of processes* $\{L(t, v) = (L^1(t, v), L^2(t, v))', \ t \geq v\}$, *nondecreasing in* t *and non increasing in* v, *such that for* $t \geq v$

$$\begin{cases} W(t) &= W(v) + \int_v^t b(X(s), W(s))ds + \int_v^t \Sigma(X(s), W(s))dB_s + (I - P')L(t, v) \\ L^1(t, v) &= \int_{s=v}^{s=t} 1_{\{W^1(s)=0\}} dL^1(s, v) \\ L^2(t, v) &= \int_{s=v}^{s=t} 1_{\{W^2(s)=0\}} dL^2(s, v). \end{cases} \tag{1.24}$$

In addition, bivariate process $Q(t) = (Q^1(t), Q^2(t))'$ *converges in distribution to* $(W^1(0), W^2(0))$ *when the queues are initially empty, i.e.* $Q(0) = (0, 0)'$. *If (1.23) does not hold convergence in distribution still holds, but the limiting distribution can be improper.*

Theorem 7 resembles Theorem 4.1 of [Ram00]. However there are some differences as [Ram00] considers the case where functions b^i depends on the i-th component of the queue level and its corresponding compensator, but, more crucially, assumes that this function is increasing with respect to that queue level (as opposed to Theorem 7, where b^i is non decreasing with respect to the queue levels other than the i-th one). Also, no diffusion is present in [Ram00].

Theorem 8 is proved by using a standard Loynes argument (see [Loy62]) which is also used in a continuous time case in dimension 1 in [SR00] and [RS03]. The main ingredient for this argument here is Theorem 7, which is a comparison theorem for (reflected) stochastic differential equations in dimension 2. The idea of proof of this result is to obtain a Gromwall estimate of the form

$$\mathbb{E}(N_A((Z_t^1 - Y_t^1)^+, (Z_t^2 - Y_t^2)^+)) \leq C \int_0^t \mathbb{E}(N_A((Z_s^1 - Y_s^1)^+, (Z_s^2 - Y_s^2)^+))ds \tag{1.25}$$

for all $t \geq 0$, for a constant $C \geq 0$, where $N_A(.)$ is the square of an appropriate norm. Since the initial conditions are such that $\mathbb{E}(N_A((Z_0^1 - Y_0^1)^+, (Z_0^2 - Y_0^2)^+)) = \mathbb{E}(N_A(0,0)) = 0$, this will imply that $\mathbb{E}(N_A((Z_t^1 - Y_t^1)^+, (Z_t^2 - Y_t^2)^+)) = 0$ for all $t \geq 0$, i.e. $Z_t^1 \leq Y_t^1$ and $Z_t^2 \leq Y_t^2$ almost surely. $N_A(x_1, x_2)$ is of the form $(x_1, x_2)A(x_1, x_2)'$ for a well chosen definite positive symmetric matrix $A \in \mathbb{R}^{2\times 2}$. This choice for A is not completely trivial and depends on matrix P. Furthermore, estimate (1.25) relies a lot (and it is not obvious at first glance) on the non decreasing properties of $b^1(x, y^1, .)$ and $b^2(x, ., y^2)$ in Assumption 2.

Under Assumption 3 a result akin to Theorem 8 holds in dimension N. Rather than quoting it in its generality, we will state such a result in a particular queueing context. Let us suppose that in (1.17) $b(.,.)$ has the form

$$b(x, y^1, ..., y^N) = \lambda(x) - (I - P')\mu(x, y^1, ..., y^N) = (b^1(x, y^1, ..., y^N), ..., b^N(x, y^1, ..., y^N))' \tag{1.26}$$

where $\lambda(.) = (\lambda^1(.), ..., \lambda^N(.))'$ has positive entries and $\mu(x, y^1, ..., y^N) = (\mu^1(x, y^1), ..., \mu^N(x, y^N))'$. We also suppose that the diffusion coefficient $\Sigma(.,.)$ is zero. This models the following behaviour : let us suppose that fluid comes from out of the network to queue i at a rate $\lambda^i(X(t))$. We suppose in this section that, for all $i = 1, ..., N$, the service rate for queue i, $\mu^i(X(t), Q^i(t))$ only depends on the queue level $Q^i(t)$. We also suppose that $\mu^i(x, .)$ is non decreasing, is Lipschitz, and verifies $\forall x, \quad \mu^i(x, 0) \leq \lambda^i(x)$. In other words, the network does not waste resources by setting a service rate larger than the exogenous arrival rate when one of the queues is empty. Then it is easy to check that $b(.,.)$ defined in (1.26) satisfies Assumption 3.

In such a context, the equivalent of Theorem 8 is the following :

Theorem 9. *Let us suppose that*

$$(I - P')^{-1}\mathbb{E}(\lambda(X(0))) < \lim_{y^1,...,y^N \to +\infty} \mathbb{E}(\mu(X(0), y^1, ..., y^N)). \tag{1.27}$$

Then there exists a non negative stationary vector valued process $\{W(t) = (W^1(t), ..., W^N(t))',\ t \geq 0\}$ such that for $t \geq v$, $W(t) = W(v) + \int_v^t b(X(s), W(s))ds$, with $b(.,.)$ defined as in (1.26). Moreover, $Q(t) = (Q_1(t), ..., Q_N(t))'$ satisfying (1.17) converges in distribution to $W(0)$ as t tends to infinity, when the network is initially empty.

If (1.27) is not satisfied then $Q(t)$ still converges in distribution, but the limiting distribution may be improper.

Let us note that when $\mu(x, y^1, ..., y^N) = \mu(x)$ is independent from the queue levels, Condition (1.27) is exactly Condition (1.3) in [KW96]. Furthermore, we underline that stability Conditions (1.23) and (1.27) are not really intuitive and weaker than the usual conditions

$$\mathbb{E}\left(\limsup_{y^1\to\infty} \sup_{y^2\geq 0} b^1(X(0), y^1, y^2)\right) < 0, \quad \mathbb{E}\left(\limsup_{y^2\to\infty} \sup_{y^1\geq 0} b^2(X(0), y^1, y^2)\right) < 0, \tag{1.28}$$

$$\mathbb{E}(\lambda(X(0))) < (I - P') \lim_{y^1,...,y^N \to +\infty} \mathbb{E}(\mu(X(0), y^1, ..., y^N)) \tag{1.29}$$

which roughly state that queues converge in distribution if, for each queue, mean drift is negative for large values of queue levels. Let us point out that (1.28) and (1.29) respectively imply (1.23) and (1.27) because matrix $(I - P')^{-1}$ has, by propriety of M- matrices, non negative entries. However the converse is not true in general.

We now present a very naive illustration of how Theorem 8 can be used in risk theory. Let us consider two branches of an insurance company $\{(R^1_t, R^2_t),\ t \geq 0\}$ with claims arriving according to a Poisson process of intensity $\alpha > 0$. We suppose that, initially, μ_1 and μ_2 are initial premium rates for branches 1 and 2. The insurance company decides to adopt the following strategy : a proportion $p_{12}\mu_1$ of premium is sent to branch 2, and a proportion $p_{21}\mu_2$ to branch 1, where p_{12} and p_{21} lie in $(0,1)$. Drifts for both branches p_i, $i = 1,2$, thus now verify, with this new strategy,

$$p := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (I - P') \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := (I - P')\mu$$

where $P = \begin{pmatrix} 0 & p_{12} \\ p_{21} & 0 \end{pmatrix}$, which will suppose is an M matrix (or, equivalently, which has a spectral radius smaller than 1). Note that this strategy is economically saving as the instantaneous premium rate for the sum of branches $R^1_t + R^2_t$ is $p_1 + p_2 \leq \mu_1 + \mu_2$, so that the difference may be e.g. invested in other assets. We suppose that claims for both branches arrive at the same time, but are independent with distribution $\mathcal{E}(\beta_1)$ (for branch 1) and $\mathcal{E}(\beta_2)$ (for branch 2). This is the situation known as *common shocks*, as will be explained later in Section 1.4. $\{R_t = (R^1_t, R^2_t),\ t \geq 0\}$ then evolves according to

$$R_t = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + (I - P')\mu t - \sum_{k=1}^{N_t} \begin{pmatrix} V^1_k \\ V^2_k \end{pmatrix}$$

for some Poisson process N_t of intensity α, $(V^1_k)_{k\in\mathbb{N}}$ i.i.d. $\sim \mathcal{E}(\beta_1)$, $(V^2_k)_{k\in\mathbb{N}}$ i.i.d. $\sim \mathcal{E}(\beta_2)$. We let

$$\mathcal{T} := \inf\{t \geq 0 |\ R_t \notin [0, +\infty)^2\} = \min(\mathcal{T}^1, \mathcal{T}^2) \quad (\text{where } \mathcal{T}^i := \inf\{t \geq 0 |\ R^i_t < 0\})$$

the exit time of the first quadrant, i.e. the minimum of the ruin times of both branches. We embed this two dimensional risk process similarly as in Section 1.4 (where things will be detailed in a more general setting) : we consider a Markov chain $\{J(t),\ t \geq 0\}$ with state space $\{0,1,2\}$ and generating matrix

$$Q = \begin{pmatrix} -\alpha & \alpha & 0 \\ 0 & -\beta_1 & \beta_1 \\ \beta_2 & 0 & -\beta_2 \end{pmatrix}.$$

The embedded process, that we again call R_t, evolves like the original process when $J(t)$ is in state 0. States 1 and 2 correspond to occurence of claims for both branches. When in state 1, R^1_t drops with rate -1 while R^2_t remains frozen ; in state 2, it is R^2_t drops with rate -1 while R^1_t remains constant. See Figure 1.3 for an illustration. Embedded process then verifies

$$\begin{cases} dR_t &= (I - P')\mu(J(t))dt - \lambda(J(t))dt, \\ R_0 &= (u^1, u^2)' \end{cases}$$

where $\mu(0) = (\mu_1, \mu_2)'$, $\mu(1) = \mu(2) = (0,0)'$, $\lambda(0) = (0,0)'$, $\lambda(1) = (1,0)'$, $\lambda(2) = (0,1)'$.

Let $\tau = \min(\tau^1, \tau^2)$ be the exit time of the embedded process out of the first quadrant. τ and

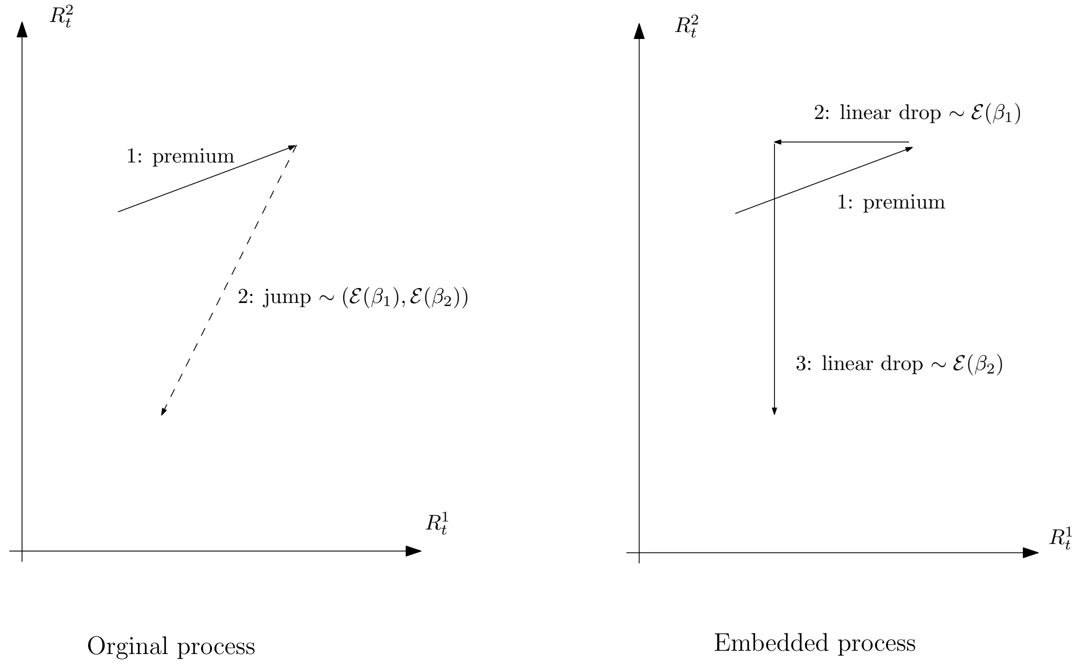

FIGURE 1.3 – Embedding.

$\mathcal{T}$ do not have the same distribution (in fact, τ is stochastically larger than $\mathcal{T}$). However we have the equivalence

$$[\tau = \min(\tau^1, \tau^2) < +\infty] \quad \Longleftrightarrow \quad [\mathcal{T} < +\infty].$$

We are interested in finding an upper bound for $\mathbb{P}(\mathcal{T} = +\infty | \, R_0 = (u^1, u^2)') = \mathbb{P}(\tau = +\infty | \, R_0 = (u^1, u^2)')$. This quantity is positive if the relative safety loading for each branch is positive, i.e. that the following holds componentwise

$$(I - P')\mu > \alpha \begin{pmatrix} 1/\beta_1 \\ 1/\beta_2 \end{pmatrix}. \tag{1.30}$$

which, as was stated before, is stronger than (1.27). Let us introduce a dual queue $\{Q(t) = (Q^1(t), Q^2(t)), \; t \geq 0\}$ associated to embedded risk process $\{R_t, \; t \geq 0\}$ satisfying

$$\begin{cases} dQ(t) &= \lambda(J^*(t))dt - (I - P')\mu(J^*(t))dt + (I - P')dL_t \\ Q(0) &= (0, 0)' \\ L^i(t) &= \int_0^t 1_{\{Q^i(s)=0\}} dL^i(s), \quad i = 1, 2, \end{cases} \tag{1.31}$$

where $\{J^*(t), \; t \geq 0\}$ is the reversed version of Markov chain $\{J(t), \; t \geq 0\}$. The following gives an upper bound for $\mathbb{P}(\mathcal{T} = +\infty | \, R_0 = (u^1, u^2)')$.

Proposition 1. $\mathbb{P}(\mathcal{T} = +\infty | \, R_0 = (u^1, u^2)')$ *verifies the following upper bound*

$$\mathbb{P}(\mathcal{T} = +\infty | \, R_0 = (u^1, u^2)') \leq \mathbb{P}(W^1 \leq u^1, W^2 \leq u^2) \tag{1.32}$$

where the corresponding dual queue $\{Q(t), \; t \geq 0\}$ *of bivariate risk process* $\{R_t, \; t \geq 0\}$ *converges* *in distribution to a finite random variable* $W = (W_1, W_2)'$.

Proof. We use Theorem 6 with $T = +\infty$, which exactly reads (1.32). □

Let us note that W depends on matrix reflection P in (1.31) in term $(I - P')dL_t$ (which happens to be the same one appearing in drift $\lambda(J^*(t)) - (I - P')\mu(J^*(t))$, for reasons explained later). By taking a matrix reflection equal to 0, it is not too difficult to see that the upper bound in (1.32) would have been replaced by $\mathbb{P}(W^{0,1} \le u^1, W^{0,2} \le u^2)$ from the middle term of (1.21) in Theorem 5 for some $W^0 = (W^{0,1}, W^{0,2})$ which is the limit in distribution of process, say $\{Q^0(t),\ t \ge 0\}$, with same drift as $\{Q(t),\ t \ge 0\}$ but with reflection matrix 0 (remember that Theorem 5 corresponds to a reflection matrix which is zero). However, this latter upper bound would not be as tight as the one in (1.32) : this due to the fact that process $\{Q^0(t),\ t \ge 0\}$ verifies $Q^0(t) \le Q(t)$ by (4.6) of Theorem 4.1 of [Ram00] (thanks to the fact that $0 \le p_{ij}$, which is exactly assumption (4.2) in that theorem).

There now remains to be convinced why upper bound in (1.32) is more easily dealt with than directly $\mathbb{P}(\mathcal{T} = +\infty |\ R_0 = (u^1, u^2)')$. It seems that studying the asymptotic stationary distribution W is an active topic. [Miy13] provides some information on the behavior of $\mathbb{P}(W^1 \le u^1, W^2 \le u^2)$ as (u^1, u^2) tends to infinity, which could provide upper asymptotic bounds for $\mathbb{P}(\mathcal{T} = +\infty |\ R_0 = (u^1, u^2)')$. Also, we did not consider the case where premium rates depend on queue level for presentation purpose, although this aspect would have been even more interesting and challenging (and more difficult).

1.3.3 Linear rates in a stochastic fluid network and case of a risk process with reinvestment

We consider in [Rab06a] the particular model (1.17) where $\{X(t),\ t \ge 0\}$ is a finite stationary Markov chain on a state space $S = \{1, ..., K\}$, $b(.,.)$ has the form (1.26) seen in the previous subsection, and where service rates are constant and equal to μ^i, $i = 1, ..., N$. Letting

$$A := (I - P')\text{diag}(\mu^1, ..., \mu^N) = (a_{ij})_{(i,j)\in\{1,...,N\}^2}, \tag{1.33}$$

then $b(x, y^1, ..., y^N)$ in (1.26) reads

$$b(x, y^1, ..., y^N) = \lambda(x) - A(y^1, ..., y^N)'.$$

Note that this model is the N dimensional counterpart to the one dimensional model (1.5) studied in Section 1.2 without noise. Solution to (1.17) is in that case $L(t) = (0, ...0)'$ (there is no reflection) and $Q(t)$ given by

$$Q(t) = \exp(-At)Q(0) + \int_0^t \exp(-A(t-s))\lambda(X(s))ds.$$

The following is the analog of Theorem 1.8 in an N-dimensional setting :

Proposition 2. $Q(t)$ *converges in distribution independently of the initial conditions to*

$$W = (W_1, ..., W_N)' := \int_{-\infty}^0 \exp(As)\lambda(X(s))ds,$$

which by making the variable change $s := -s$ is

$$W = \int_0^{\infty} \exp(-As)\lambda(X^*(s))ds. \tag{1.34}$$

The remaining of this subsection is devoted to finding joint moments of the W_i's. We define the joint Laplace transform of W given $X^*(0)$: $\forall i \in S$, $\forall u = (u_1, ..., u_N)' \in (-\infty, 0]^N$,

$$\phi_i(u) := \mathbb{E}(\exp(u'W)|X^*(0) = i) = \mathbb{E}\left(\exp\left(\sum_{k=1}^{N} u_k W_k\right) \middle| X^*(0) = i\right)$$

and we set $\phi(u) := (\phi_1(u), ..., \phi_K(u))'$. In order to avoid too cumbersome notation we will let $\mathcal{N} := \{1, ..., N\}$. We also set

$$\forall l \in S \quad \Lambda(l) := \text{diag }(\lambda^1(l), ..., \lambda^N(l)).$$

In the following we will use the following notation for indices in $\mathcal{N}^n$

$$\underline{l}_n := (l_1, ... l_n), \quad \underline{k}_n := (k_1, ... k_n).$$

By a classical renewal argument, one has the following differential equation for $\phi(.)$ (remember that Q^* is the matrix generator of reversed Markov chain $\{X^*(t),\ t \in \mathbb{R}\}$) :

Proposition 3. *ϕ satisfies the following differential equation for $u \in (-\infty, 0]^N$:*

$$\nabla\phi(u)\ A'u = (F(u) + Q^*)\phi(u) \tag{1.35}$$

where $F(u) := diag\ (u'\lambda(1), ..., u'\lambda(K))$, and $\nabla\phi(u)$ is the $K \times N$ matrix of which (i, j)th element is $\partial_j\phi_i(u)$, $i \in S$, $j \in \mathcal{N}$ (gradient of ϕ).

Proving Proposition 3 is done in two steps. First, one proves by a classical renewal argument that $\phi(u)$ satisfies some integral equation of the form

$$\phi(u) = \int_0^{\infty} \chi_\nu(u, x)\phi(\exp(-A'x)u)dx$$

for a smooth enough $\chi_\nu : (-\infty, 0]^N \times [0, +\infty) \longrightarrow \mathbb{R}^{K\times K}$, and where ν is some large enough positive parameter, issued from uniformization of Markov chain $\{X^*(t),\ t \in \mathbb{R}\}$. This in particular entails that $\phi(.)$ is (infinitely) differentiable. Then one obtains (1.35) by manipulating the above integral equation and letting $\nu \to +\infty$.

We aim at determining recursively the following quantities

$$m_i^n(\underline{l}_n) = m_i^n(l_1, ..., l_n) := \mathbb{E}(W_{l_1}...W_{l_n}|X^*(0) = i), \quad i \in S,\ \underline{l}_n \in \mathcal{N}^n$$

from which joint moments $\mathbb{E}(W_1^{n_1}...W_N^{n_N}|X^*(0) = i)$, $n_1, ..., n_N \in \mathbb{N}^N$, are available. We set

$$\begin{aligned} m^n(\underline{l}_n) &:= (m_1^n(\underline{l}_n), ..., m_K^n(\underline{l}_n))' \in \mathbb{R}^{K\times 1}, \\ m^n &:= \{m^n(\underline{l}_n),\ \underline{l}_n \in \mathcal{N}^n\}. \end{aligned}$$

m^n is thus a family of column vectors indexed by $\mathcal{N}^n$, with $m^0 := (1, ..., 1)' \in \mathbb{R}^{K\times 1}$. We also introduce

- $A^{(n)} = \left(a^{(n)}_{\underline{l}_n,\underline{k}_n}\right)_{(\underline{l}_n,\underline{k}_n)\in\mathcal{N}^n\times\mathcal{N}^n}$ the $N^n \times N^n$ matrix defined by

$$a^{(n)}_{\underline{l}_n,\underline{k}_n} = \begin{cases} \sum_{i=1}^n a_{l_i l_i} & \text{if } \underline{l}_n = \underline{k}_n \\ a_{l_i k_i} & \text{if } l_j = k_j,\ j \neq i, \text{ and } l_i \neq k_i \\ 0 & \text{otherwise.} \end{cases} \tag{1.36}$$

- for all $n \in \mathbb{N}$ the $(N^n \times K) \times (N^{n-1} \times K)$ block matrix by

$$\Lambda^{(n)} = \left(b^{(n)}_{\underline{l}_n,\underline{k}_{n-1}}\right)_{(\underline{l}_n,\underline{k}_{n-1}))\in\mathcal{N}^n\times\mathcal{N}^{n-1}}$$

where each $b^{(n)}_{\underline{l}_n,\underline{k}_{n-1}}$ is a $K \times K$ matrix defined by

$$b^{(n)}_{\underline{l}_n,\underline{k}_{n-1}} := \begin{cases} \text{diag}\left(\lambda^{k_i}(1), ..., \lambda^{k_i}(K)\right) & \text{if } l_p = k_p,\ p = 1, ..., i-1, \\ & \text{and } l_{p+1} = k_p,\ p = i, ..., n-1 \\ 0 & \text{otherwise.} \end{cases}$$

- Finally, we denote by I_S and $I_{\mathcal{N}^n}$ respectively the $K \times K$ and $N^n \times N^n$ identity matrices.

Before stating the result that gives relation between m^n and m^{n-1}, we first observe that (1.35) reads for each $j \in S$

$$\sum_{p\in\mathcal{N}} \left(\sum_{k\in\mathcal{N}} a_{kp} u_k\right) \partial_p \phi_j(u) = \left(\sum_{i\in\mathcal{N}} u_i \lambda^i(j)\right) \phi_j(u) + \sum_{k\in S} q^*_{jk} \phi_k(u). \tag{1.37}$$

Differentiating the above with respect to l_i, $i = 1, ..., n$, and evaluating at $u = (0, ..., 0)'$, we then arrive at

$$\begin{aligned} \sum_{p\in\mathcal{N}} \sum_{i=1}^n a_{l_i p} m^n_j(l_1, ..., l_{i-1}, p, l_{i+1}, ..., l_n) \quad &= \quad \sum_{i=1}^n \lambda^{l_i}(j) m^{n-1}_j(l_1, ..., l_{i-1}, l_{i+1}, ..., l_n) \\ &\quad + \sum_{k\in S} q^*_{jk} m^n_k(l_1, ..., l_n) \end{aligned}$$

for all $\underline{l}_n = (l_1, ..., l_n)$, which reads in a more compact form

$$\begin{aligned} & \left(A^{(n)} \otimes I_S\right) m^n = \Lambda^{(n)} m^{n-1} + \left(I_{\mathcal{N}^n} \otimes Q^*\right) m^n \\ \Longleftrightarrow \quad & \left(A^{(n)} \otimes I_S - I_{\mathcal{N}^n} \otimes Q^*\right) m^n = \Lambda^{(n)} m^{n-1} \end{aligned} \tag{1.38}$$

where we recall that if $M = (m_{ij})$ is a $d \times d$ matrix and N is a $p \times p$ matrix, $M \otimes N \in \mathbb{R}^{dp\times dp}$ is the Kronecker product of matrices M and N.

One then sees that, provided that the $(N^n K) \times (N^n K)$ matrix

$$M^{(n)} := A^{(n)} \otimes I_S - I_{\mathcal{N}^n} \otimes Q^*$$

is invertible, then one gets from (1.38) an expression of m^n in function of m^{n-1}. This is in fact not obvious, and is precisely the point of the following result, which states that this is true under certain hypothesis on matrix A :

Theorem 10. *Suppose that A verifies*

$$(\mathbf{C}1)\begin{cases} \forall i = 1, ..., N, & \sum_{j=1}^{n} a_{ij} \geq 0, \\ \exists i_0 \in \{1, ..., N\}, & \sum_{j=1}^{n} a_{i_0 j} > 0. \end{cases}$$

Then $A^{(n)} \otimes I_S - I_{\mathcal{N}^n} \otimes Q^*$ is invertible and $(m^n)_{n \in \mathbb{N}}$ is determined recursively by the relation

$$m^n = \left(A^{(n)} \otimes I_S - I_{\mathcal{N}^n} \otimes Q^*\right)^{-1} \Lambda^{(n)} m^{n-1} \tag{1.39}$$

with $m^0 = (1, ..., 1)'$.

We recall that A is defined by (1.33). It is easy to check that A' satisfies $(C1)$. This is due to the fact that matrix P is stochastic and verifies (1.18). However, A does not necessarily satisfisfies this condition. The general case will be dealt with in a subsequent result, but we are first going to give some elements of proof of Theorem 10. We first say that some square matrix R satisfies $(C2)$ if

(C2) • R satisfies **(C1)**,
- R is of the form $R = D_R - C_R$ where D_R is a diagonal matrix with positive diagonal elements, and C_R is a matrix with 0's on its diagonal and non-negative off-diagonal elements,
- C_R is irreducible (in the sense that its corresponding directed graph is strongly connected, see Definition 2 p.50 of [Gan66]).

The following lemma is a consequence of Lemma 2.2.1 in [Neu81] :

Lemma 2. *Let R be a matrix satisfying* **(C2)***. Then R is invertible.*

One then first proves that $A^{(n)}$ satisfies $(C2)$, then that $A^{(n)} \otimes I_S$ also satisfies this condition. Thus it has a decomposition of the form $A^{(n)} \otimes I_S = D_{A^{(n)} \otimes I_S} - C_{A^{(n)} \otimes I_S}$ as explained in Condition $(C2)$. Since Q^* is the generating matrix of a Markov chain, $I_{\mathcal{N}^n} \otimes Q^*$ is a matrix of negative diagonal elements and non-negative off-diagonal elements, of which sums on each row add up to zero. This implies that $M^{(n)}$ has a decomposition of the form

$$M^{(n)} = D_{M^{(n)}} - C_{M^{(n)}}.$$

The final step consists in veryfying that $M^{(n)}$ verifies $(C1)$ and that $C_{M^{(n)}}$ is irreducible, which implies that it is invertible thanks to Lemma 2.

We now turn to the case where A does not verify $(C1)$. The trick is to notice that $A' = D_\mu(I - P)$ satisfies $(C2)$, then to write A' in the form

$$A' = J - L$$

where J is a diagonal matrix and, most importantly, L is the generating matrix of an irreducible stationary Markov chain say $\{Y(t),\ t \in \mathbb{R}\}$. If we let $(\nu_1, ..., \nu_K)$ its distribution and $H :=$

diag $(\nu_1, ..., \nu_N)$, then $L^* := H^{-1}L'H$ is the generating matrix of its reversed version $\{Y^*(t) := Y((-t)^+),\ t \in \mathbb{R}\}$. One can check that this in particular implies that

$$\hat{A} := J - L^*$$

verifies $(C2)$. Since J and H are diagonal matrices, one has that

$$\hat{A} = H^{-1}AH \quad \Longleftrightarrow \quad A = H\hat{A}H^{-1}.$$

Let us set $\bar{\lambda}(.) := H^{-1}\lambda(.)$, and consider

$$\begin{aligned} \hat{W} &:= \int_0^\infty \exp(-\hat{A}s)H^{-1}\lambda(X^*(s))ds = \int_0^\infty \exp(-\hat{A}s)\bar{\lambda}(X^*(s))ds \\ &= H^{-1}W. \end{aligned}$$

Since $\hat{A}$ satisfies $(C2)$, one can obtain all joint moments of $\hat{W}$ by replacing A by $\hat{A}$ and $\lambda(.)$ by $\bar{\lambda}(.)$ in Theorem 10. This yields joint moments of W thanks to relation $W = H\hat{W}$.

To finish this subsection, and as in subsection 1.3.2, we link how these results can be applied to the 2 dimensional simple continuous ruin model (1.19) which here reads

$$\begin{cases} dR_t &= AR_t dt - \lambda(X^*(t))dt \\ R_0 &= (u^1, u^2)'. \end{cases} \tag{1.40}$$

This is a model with two branches R_t^1, R_t^2, where claims (modelled by $\lambda(X^*(s))dt$) are paid to subscribers at a continuous (modulated) rate. The most interesting part in (1.40) is that each branch $i \in \{1,2\}$ is continuously reinvested at rate μ^i, but also that funds are transferred from branch i to branch j at rate $p_{ji}\mu^j R_t^j$. Thus we have a simple 2 dimensional risk process with transactions, however without premium rate, as in the 1 dimensional application (1.6). As explained in Theorem 5, ruin probability $\psi(u,+\infty) = \mathbb{P}(\nu < +\infty |\ R_0 = u)$ (where definition of ruin time ν is explained in Section 1.3.1) is related to the stationary limiting random variable $W \stackrel{\mathcal{D}}{=} \lim_{t\to\infty} Q(t)$ where $\{Q(t) = (Q^1(t), Q^2(t))',\ t \geq 0\}$ is defined along with its compensator $\{L_t = (L_t^1, L_t^2)',\ t \geq 0\}$ by the reflected equation

$$\begin{cases} dQ(t) &= \lambda(X(t))dt - AR_t dt + dL_t \\ L_t^i &= \int_0^t 1_{\{Q^i(s)=0\}} dL_s^i, \quad i = 1, 2, \\ Q_0 &= (0,0)'. \end{cases} \tag{1.41}$$

where matrix reflection here is $P = 0$ with notation of Section 1.3.1. As recalled in the beginning of the present section, the compensator is $L_t \equiv (0,0)'$, so that W is given by (1.34) in Proposition 2. By inequality (1.21) in Theorem 5 with $T = +\infty$, $W = (W^1, W^2)'$ is thus such that

$$\psi(u,+\infty) \leq \mathbb{P}(W^1 \geq u^1, W^2 \geq u^2), \tag{1.42}$$

so bounds for $\psi(u,+\infty)$ are available thanks to moments of $W = (W^1, W^2)'$ computed in this section. Note that point $(0,0)'$ has a special role for risk process $\{R_t,\ t \geq 0\}$. It turns out that it is the summit of cone $\mathcal{C} := \{(r_1, r_2) \in \mathbb{R}^2 |\ \mu^1 r_1 - p_{21}\mu^2 r_2 < 0,\ \mu^2 r_2 - p_{12}\mu^1 r_1 < 0\} = \{(r_1, r_2) \in \mathbb{R}^2 |\ r_2 > \frac{\mu^1}{\mu^2}\frac{1}{p_{21}} r_1,\ r_2 < \frac{\mu^1}{\mu^2} p_{12} r_1\} \subset \mathbb{R}^2$ illustrated in Figure 1.4. Besides, it can be checked that $\mathcal{C}$ is *absorbing* for process $\{R_t,\ t \geq 0\}$ if direction of claims $-\lambda(i)$, $i = 1, ..., K$, is included in cone

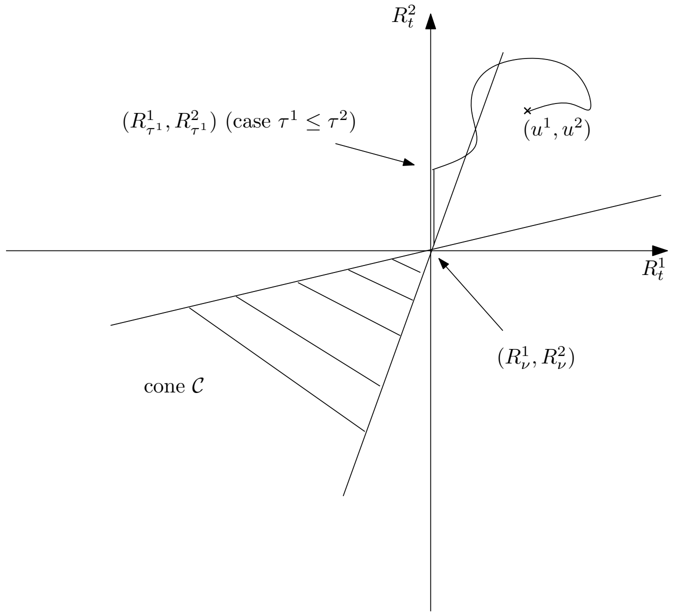

FIGURE 1.4 – Cone $\mathcal{C}$ and sample path up to ruin, no premium.

$\mathcal{C}$. In other words, risk process $\{R_t,\ t \geq 0\}$ enters $\mathcal{C}$ through its summit and $(0,0)'$ can be seen as a "definitive ruin threshold" for this model without premium. This terminology will be used in the forthcoming Section 1.4.

One might wonder what kind of result one has if one introduces premium rates for R_t. Let us then suppose that it satisfies, instead of (1.40),

$$\begin{cases} dR_t &= pdt + AR_t dt - \lambda(X^*(t))dt \\ R_0 &= (u^1, u^2)' \end{cases} \tag{1.43}$$

where $p = (p_1, p_2)'$ are premium rates, with $p_i > 0$, $i = 1, 2$. Remembering that $A = (I - P')\text{diag}(\mu^1, \mu^2)$, the trick is to set $\tilde{R}_t := R_t + \text{diag}(1/\mu^1, 1/\mu^2)(I - P')^{-1}p$. Then $\tilde{R}_t$ satisfies (1.40) with a different initial condition $\tilde{R}_0 = \tilde{u} = (\tilde{u}^1, \tilde{u}^2)' = (u^1, u^2)' + \text{diag}(1/\mu^1, 1/\mu^2)(I - P')^{-1}p$. The ruin probability ν of $\tilde{R}_t$ verifies inequality analog to (1.42)

$$\mathbb{P}(\nu < +\infty |\ \tilde{R}_0 = \tilde{u}) \leq \mathbb{P}(W^1 > \tilde{u}^1, W^2 > \tilde{u}^1).$$

Ruin probability ν for $\tilde{R}_t$ is interpreted for R_t as follows : it corresponds to first hitting time of R_t, starting from $(u^1, u^2)'$, of point

$$(s_1^*, s_2^*)' := -\text{diag}(1/\mu^1, 1/\mu^2)(I - P')^{-1}p \in (-\infty, 0)^2.$$

As for the case without premium, $(s_1^*, s_2^*)'$ can also be seen as a "definitive ruin threshold", see Figure 1.5.

A future work is to complexify model and either add modulation for matrix A or replace continuous decreasing process $-\lambda(X^*(s))dt$ by a negative jump process dS_t. Adding a premium rate to (1.40), as explained in section 1.2, amounts to consider the time of absolute ruin, rather than time of ruin. This is precisely the point of the following section.

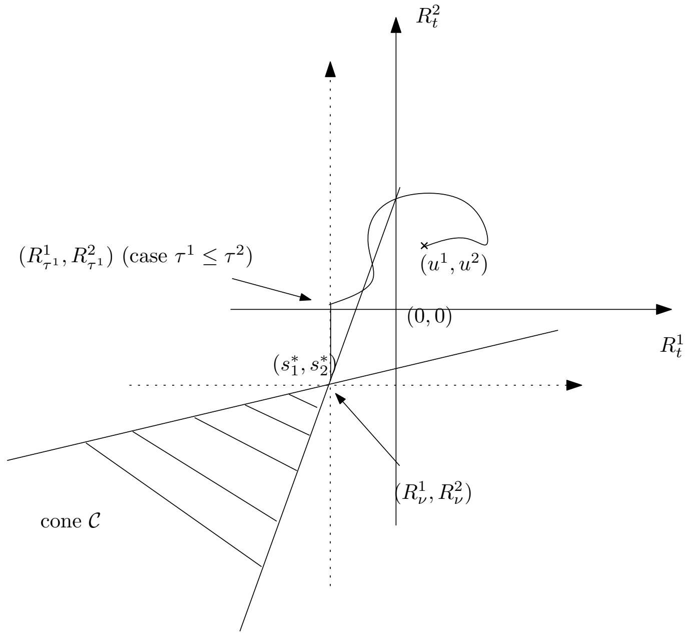

FIGURE 1.5 – Cone $\mathcal{C}$ and sample path up to ruin, with premium.

1.4 A particular multidimensional risk model with common shocks

We present here the part of [Rab09] dealing with risk processes in dimension larger than 1 with reinvestment. Note that this corresponds to model (1.40) studied in the previous subsection without injection to one capital from the other branches, but with premium rates and claim occurrences modelled by a jump process. We first start by giving results for a certain risk process in dimension 1 associated to its dual fluid queue, then explain, by again a duality argument, how one can pass from results related to this fluid queue to the multidimensional ruin problem.

1.4.1 Risk processes with reinvestment and absolute ruin.

We consider the following risk process

$$\begin{cases} dR_t &= \delta(\Phi(t))R_t dt + c(\Phi(t))dt - d\mathcal{S}(t) \\ R_0 &= x. \end{cases} \tag{1.44}$$

Notation differ slightly from the previous sections :

- The process $\{\Phi(t),\ t \geq 0\}$ is an underlying finite continuous time stationary irreducible Markov chain of state space denoted by E_Φ, describing the state of the environment, and of infinitesimal matrix M. We let $\pi = (\pi_i,\ i \in E_\Phi)$ be its stationary distribution.
- $\mathcal{S}(t)$ is the aggregate claim amount at time t. It is a non decreasing, piecewise constant process, of which jumps occur when a separate independent irreducible stationary Markov chain $\Phi^{\mathcal{S}}(t)$ reaches certain states. Given the state of the underlying Markov chain $\Phi(t) = i$ at time t, its jumps follow a Phase type distribution $PH(\gamma_i, G_i, t_i)$, as defined in the beginning of Section 1.1.2.
- $\delta(\Phi(t)) > 0$ is the instant rate of credit interest when the surplus $R_t > 0$, and the instant

rate of debit interest when $R_t \leq 0$ (see e.g. [CGY06] for details).
– $c(\Phi(t))$ is the premium rate.

In other words, process (1.44) is one dimensional risk process with Phase type distributed inter-claims, Markov modulated reinvestment and premium rate, and Phase type distributed claims (of which distribution is also modulated). According to SubSection 1.1.2, the corresponding embedded process, also denoted by $\{R_t,\ t \geq 0\}$ verifies

$$\begin{cases} dR_t &= \delta(\varphi(t))R_t dt - h(\varphi(t))dt \\ R_0 &= x, \end{cases} \tag{1.45}$$

for an extended Markov chain $\{\varphi(t),\ t \in \mathbb{R}\}$ of extended state space E_φ, with infinitesimal generator matrix T. We recall that a parameter $a > 0$ has to be taken into account in embedding (1.45), although dependence of R_t on a is not mentioned.

Contrarily to the previous sections, we are here going to study the *time of absolute ruin* defined by

$$\mathcal{T}_*(x) := \inf\{s \geq 0 \mid R_s < s^*\},$$

and its counterpart for the embedded process $\tau_{*a}(x)$ (here dependence on a is recalled), where $s^* < 0$ is the definitive ruin threshold defined by

$$s^* = \min\{-c(i)/\delta(i),\ i \in E_\Phi\}.$$

This definitive ruin threshold is justified by the fact that, once below this value, process R_t can never recover and can no longer return to set $[s^*, +\infty)$, see e.g. [GY07]. The corresponding dual queueing process is in that case

$$\begin{cases} dQ_t &= -\delta(\varphi^*(t))Q_t dt + h(\varphi^*(t))dt \\ Q_0 &= s^*. \end{cases} \tag{1.46}$$

Q_t is here reflected at s^*, not 0, because we are interested in the first passage time below s^*. Let us note that in that case corresponding compensator $\{L_t,\ t \geq 0\}$ is zero as process $\{Q_t,\ t \geq 0\}$ defined in (1.46) is always larger than s^*.

In view of introducing distributions of $\mathcal{T}_*(x)$ and $\tau_{*a}(x)$, we let

$$\begin{aligned} F_i(t,x) &= \mathbb{P}(\tau_*(x) < t,\ \varphi(0) = i), \quad \text{for } i \in E_\varphi \\ \mathcal{F}_i(t,x) &= \mathbb{P}(\mathcal{T}_*(x) \leq t,\ \Phi(0) = i) \quad \text{for } i \in E_\Phi \end{aligned}$$

the cdf of (absolute) ruin times of respectively the original and embedded process. The reason why one is interested in $\tau_{*a}(x)$ and not just in $\mathcal{T}_*(x)$ is that $\tau_{*a}(x)$ accounts for the lump of money $\mathcal{V}_*(x)$ that the insurance company can pay up to (absolute) ruin. This means that $\mathcal{S}(\mathcal{T}_*(x)) - \mathcal{V}_*(x)$ is the shortage of money that has to be paid in order to credit the very last claim that caused ruin, i.e. the *ruin severity*. Free parameter $a > 0$ enables to consider joint Laplace transform of $\mathcal{T}_*(x)$ and $\mathcal{V}_*(x)$; this is done in the following result :

Proposition 4. *The Laplace transform* $\psi_{*i}(u,v) := \mathbb{E}\left(\exp(-u\mathcal{T}_*(x) - v\mathcal{V}_*(x))1_{\{\Phi(0)=i\}}\right)$ *of the joint distribution of* $(\mathcal{T}_*(x), \mathcal{V}_*(x))$*,* $x \geq 0$*, and* $\Phi(0) = i$*,* $i \in E_\Phi$*, verifies*

$$\psi_{*i}(u,v) = \frac{\pi_i^a}{\pi_i} \int_0^\infty e^{-t} g_{*i}(v/u, dt/u)$$

for all $u, v > 0$, *where* $g_{*i}(a, t) = \mathbb{P}(\tau_{*a}(x) \leq t,\ \varphi(0) = i)$, $x \geq s^*$.

Duality between the embedded risk and queueing process is given by the analog of Theorem 1 :

Proposition 5. *For all* $t \geq 0$, $x \geq s^*$ *and* $i \in E_\varphi$, $\mathbb{P}(\tau_*(x) < t,\ \varphi(0) = i) = \mathbb{P}(Q_t > x,\ \varphi^*(t) = i)$.

We present results concerning the Laplace transforms in x of $x \mapsto (F_i(t, x))_{i \in E_\varphi}$ and $x \mapsto (\mathcal{F}_i(t, x))_{i \in E_\Phi}$. We introduce

$$c_k(t, i) := \int_{s^*}^{+\infty} x^k F_i(t, x) dx, \quad c_k(t) := (c_k(t, i),\ i \in E_\varphi), \quad k \in \mathbb{N}. \tag{1.47}$$

It can be proved that expression of the $c_k(t)$ is given recursively in the form

$$c_k(t) = \left(\int_0^t (c_{k-1}(u) k H + \Delta_k(u)) e^{-(T^* - (k+1) D_\delta) u} du \right) e^{(T^* - (k+1) D_\delta) t}, \quad k \geq 1, \tag{1.48}$$

where T^* is the matrix generator of reversed Markov process $\{\varphi^*(t),\ t \geq 0\}$, $D_\delta = \text{diag}(\delta(i),\ i \in E_\varphi)$, $H = \text{diag}(h(i),\ i \in E_\varphi)$, and for some matrix valued function $\Delta_k(.)$. The Laplace transform of $x \mapsto (F_i(t, x))_{i \in E_\varphi}$ is then denoted by

$$\Psi_*(t, \theta) := \int_{s^*}^{\infty} e^{\theta x} F(t, x) dx = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} c_k(t). \tag{1.49}$$

We note that, thanks to Fubini and duality relation in Proposition 5, one can check that $\Psi_*(t, \theta)$ is related to Laplace transform of the corresponding dual queue level $\mathbb{E}(e^{\theta Q_t})$ via

$$\theta \Psi_*(t, \theta) = \mathbb{E}(e^{\theta Q_t} 1_{\{\varphi^*(t) = .\}}) - e^{\theta s^*} \pi^a. \tag{1.50}$$

A similar approach yields an expression of Laplace transform of $x \in [s^*, +\infty) \mapsto (\mathcal{F}_i(t, x))_{i \in E_\Phi}$

$$\int_{s^*}^{\infty} e^{\theta x} \mathcal{F}(t, x) dx$$

which we will suppose has some closed expression. We will also suppose from now on that, thanks to an inverse Laplace transform, $F_i(t, x)$ and $\mathcal{F}_i(t, x)$ are available for all i, t and $x \geq s^*$, although one is aware that, from a practical point of view, efficient and computationally fast inversion of Laplace transform is not always easy to perform.

Another aspect mentioned in [Rab09], is that double Laplace tranforms in x and t of distribution of first passage times below zero

$$\mathcal{T}(x) = \inf\{s \geq 0 \mid R_s < 0\}$$

and $\tau_a(x)$ (for the embedded process) are available, only when jumps are exponentially distributed with same parameter, and when Markov chain $\Phi(t)$ has a special structure (for example, when interclaims have same Phase type distribution). This again rises the computational issue of inverting the Laplace Transform to get e.g. the cumulative distibution function of $\mathcal{T}(x)$.

1.4.2 The $\mathbb{R}^K$ valued risk process.

We now consider a K-dimensional analog of (1.45). Let $\{R_t = (R_t^1, ..., R_t^K),\ t \geq 0\}$ be a process , of which R_t^i will be referred to as the i-th branch, satisfying

$$\begin{cases} dR_t &= \delta(\Phi(t))R_t dt + c(\Phi(t))dt - d\mathcal{S}(t) \\ R_0 &= x = (x_1, ..., x_K)' \in \mathbb{R}^K. \end{cases} \tag{1.51}$$

where $\{\Phi(t),\ t \geq 0\}$ is a modulating Markov chain, $c_i(\Phi(t))$, $\delta(\Phi(t))$ and $\mathcal{S}_i(t)$ respectively are the premium rate, interest rate (which is identical for all branches) and the total claim amount at time t for the ith branch. Claims are assumed Phase type distributed, with possible correlations. As in the one dimensional case, one objective is to embed this risk process into a continuous one. However, there are several ways of doing this embedding, that vary in function of structure of the K-dimensional jump process $\{\mathcal{S}(t),\ t \geq 0\}$. We will detail the construction when all processes $\mathcal{S}_i(t)$ jump at the same time. This means that claims occur at the same moment, when they do occur ; in particular this includes the case where a claim amount is equal to zero (i.e., no claim) for one branch and positive for another.

The embedding process, again denoted by $\{R_t = (R_t^1, ..., R_t^K),\ t \geq 0\}$, is as follows : at the time of occurrence of a claims, the vertical jump due to the claim (if any) for branch 1 is replaced by an oblique line with slope $-1/a_1$ where $a_1 > 0$ is a free parameter. In the meantime, all other branches are "frozen". Then, vertical jump due to the claim for branch 2 is replaced by an oblique line with slope $-1/a_2$ and other branches are frozen, and so on. This construction is graphically illustrated in Figures 1.6 and 1.7. The continuous embedded process then verifies the analog of

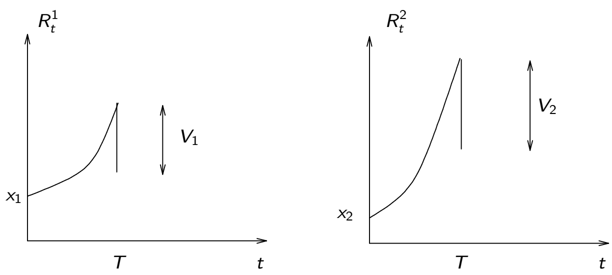

FIGURE 1.6 – Original risk processes

(1.45) :

$$\begin{cases} dR_t &= \delta(\varphi(t))R_t dt - h(\varphi(t))dt \\ R_0 &= x = (x_1, ..., x_K)' \in \mathbb{R}^K \end{cases} \tag{1.52}$$

where $\varphi(t)$ is, as in subsection 1.4.1, an extension of the initial Markov chain $\Phi(t)$ that depends on several positive free parameters a_1,...,a_K, and $h(\varphi(t)) = (h_1(\varphi(t)), ..., h_K(\varphi(t)))$ is now a K-dimensional process. For each process, one can define as in subsection 1.45 the definitive ruin threshold $s_k^* = \min\{-c_k(i)/\delta(i),\ i \in E_\Phi\}$, $k = 1, ..., K$, and times of absolute ruin $\tau_{*a_k}^k(x)$, $k = 1, ..., K$ for each branch. The goal of this section is study joint distribution of

$$(\tau_{*a_1}^1(x), ..., \tau_{*a_K}^K(x)),$$

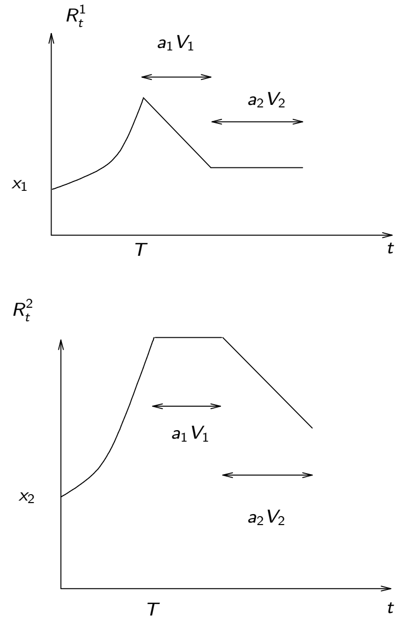

FIGURE 1.7 – Corresponding embedded risk processes

and in particular in the exit time $\tau_*(x) = \tau_{*a_1,\dots,a_K}(x)$ out of domain $\mathcal{D}$ defined as

$$\begin{aligned} \mathcal{D} &= \prod_{i=1}^{K} [s_i^*, +\infty) \subset \mathbb{R}^K, \\ \tau_*(x) &= \inf\{t \geq 0 |\ R_t \notin \mathcal{D}\} = \min(\tau^1_{*a_1}(x), \dots, \tau^K_{*a_K}(x)). \end{aligned}$$

Since the s_i^*'s are definitive ruin thresholds, $\mathcal{D}$ can be seen as a *recovery zone* for the multidimensional risk process, which means that, even if one of the branch is negative (i.e. R_t exits set $[0, +\infty)^K$), then it can always (with positive probability) become solvable again later on so long as it stays in $\mathcal{D}$. See Figure 1.8 for a sample path. Also note that $\mathbb{R}^K \backslash \mathcal{D}$ is an *absorbing set* for $\{R_t = (R_t^1, \dots, R_t^K),\ t \geq 0\}$. The exit time out of $\mathcal{D}$ of the original (not embedded) risk process is denoted by $\mathcal{T}_*(x)$. Similarly to Proposition 4 in the one dimensional case, the point of working with $\tau_*(x) = \tau_{*a_1,\dots,a_K}(x)$ and not $\mathcal{T}_*(x)$, is that free parameters $a_1, \dots, a_K$ allow for some flexibility and yield joint distribution of $\mathcal{T}_*(x)$ and the amount of money $\mathcal{V}_{*i}(x)$ ($i = 1 \dots K$) paid by each branch upon exiting $\mathcal{D}$:

Proposition 6. *Let $x = (x_1, \dots, x_K) \in \mathcal{D}$. The Laplace transform*

$$\psi_{*i}(u, v_1, \dots, v_K) = \psi_{*i}(x, u, v_1, \dots, v_K) := \mathbb{E}\left(\exp(-u\mathcal{T}_*(x) - v_1\mathcal{V}_{*1}(x) - \dots - v_K\mathcal{V}_{*K}(x))1_{\{\Phi(0)=i\}}\right)$$

of the joint distribution of $(\mathcal{T}_*(x), \mathcal{V}_{*1}(x), \dots, \mathcal{V}_{*K}(x))$, $x \geq 0$, and $\Phi(0) = i$, $i \in E_\Phi$, verifies

$$\psi_{*i}(u, v_1, \dots, v_K) = \frac{\pi_i^{v_1/u,\dots,v_K/u}}{\pi_i} \int_0^\infty e^{-t} g_i(v_1/u, \dots, v_K/u, dt/u)$$

for all $u, v_1, \dots, v_K > 0$, where $g_i(a_1, \dots, a_K, t) = \mathbb{P}(\tau_{*a_1,\dots,a_K}(x) \leq t,\ \varphi(0) = i)$.

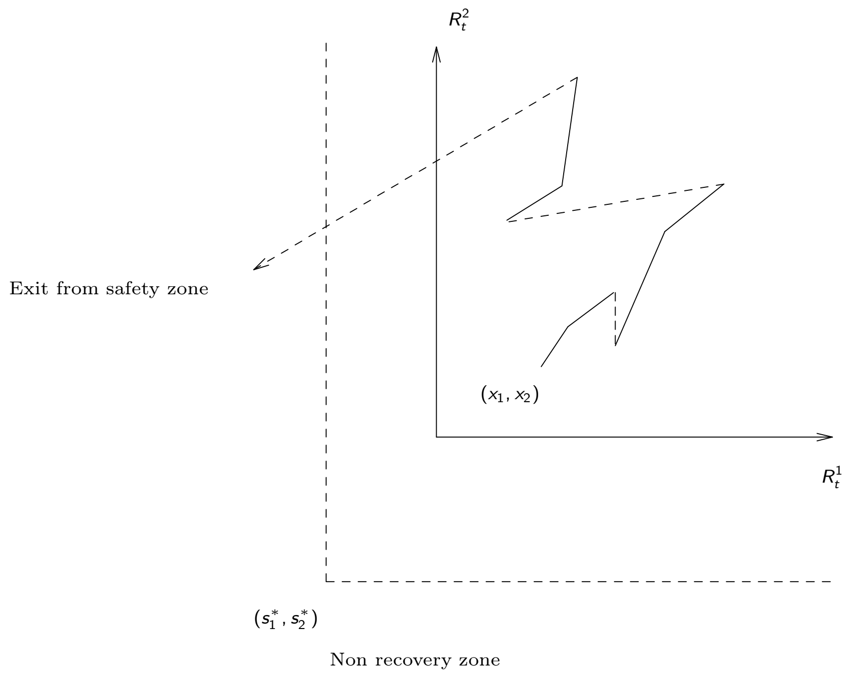

FIGURE 1.8 – Sample path of the multidimensional risk process

The remaining of this section is devoted to determining the Laplace transform in $x = (x_1, ..., x_K)' \in \mathcal{D}$ of quantity $\mathbb{P}(\tau_{*a_1,...,a_K}(x) \le t,\ \varphi(0) = i)$, i.e.

$$\int_{x\in\mathcal{D}} e^{v'x}\mathbb{P}(\tau_{*a_1,...,a_K}(x) \le t,\ \varphi(0) = i)dx, \quad v = (v_1, ..., v_K)' \in [0, +\infty)^K,\ i \in E_\Phi \tag{1.53}$$

so that in turn it will be possible from Proposition 6 to get the Laplace transform in x of $\psi_{*i}(x, u, v_1, ..., v_K)$:

$$\int_{x\in\mathcal{D}} e^{v'x}\psi_{*i}(x, u, v_1, ..., v_K)dx, \quad v = (v_1, ..., v_K)' \in [0, +\infty)^K,\ i \in E_\Phi.$$

Again, we recall that a subsequent issue is to practically invert these multivariate Laplace transforms. Getting back to (1.53), one has, thanks to the inclusion-exclusion formula :

$$\begin{aligned}\mathbb{P}(\tau_{*a_1,...,a_K}(x) \le t,\ \varphi(0) = i) &= \mathbb{P}\left(\bigcup_{j=1}^{K}[\tau_*^j(x_j) < t],\ \varphi(0) = i\right)\\ &= \sum_{I\subset\{1,...,K\}} (-1)^{\mathrm{Card}(I)+1}\mathbb{P}\left(\bigcap_{j\in I}[\tau_*^j(x_j) < t],\ \varphi(0) = i\right)\\ &= \sum_{I\subset\{1,...,K\}} (-1)^{\mathrm{Card}(I)+1}\mathbb{P}\left(A_t^I,\ \varphi(0) = i\right) \end{aligned} \tag{1.54}$$

where $A_t^I = A_t^I(x) := \bigcap_{j\in I}[\tau_*^j(x_j) < t]$, so that computing (1.53) amounts from (1.54) to find

$$\int_{x\in\mathcal{D}} e^{v'x}\mathbb{P}\left(A_t^I(x),\ \varphi(0)=i\right)dx. \tag{1.55}$$

Step 1 : a duality result. As in Proposition 5, one has the following duality between embedded process $\{R_t = (R_t^1, ..., R_t^K),\ t\geq 0\}$ verifying (1.52) and a K-dimensional dual queue $\{Q_t = (Q_t^1, ..., Q_t^K),\ t\geq 0\}$ that verifies

$$\left\{\begin{array}{rcl} dQ_t &=& -\delta(\varphi^*(t))Q_t dt + h(\varphi^*(t))dt \\ Q_0 &=& s^* = (s_1^*, ..., s_K^*)'. \end{array}\right. \tag{1.56}$$

Proposition 7. *For all $t\geq 0$, $x\in\mathcal{D}$, $I\subset\{1,...,K\}$ and $i\in E_\varphi$, $\mathbb{P}(A_t^I,\ \varphi(0)=i) = \mathbb{P}(C_t^I,\ \varphi^*(t)=i)$, where $C_t^I = C_t^I(x) := \bigcap_{j\in I}[Q_t^j > x_j]$.*

Step 2 : the dual queue. We introduce for all $v=(v_1,...,v_K)'\in[0,+\infty)^K$

$$q_t^v := v'Q_t.$$

Since Q_t satisfies the linear equation (1.56), it easy to check that $\{q_t^v,\ t\geq 0\}$ verifies the following one dimensional linear equation

$$\left\{\begin{array}{rcl} dq_t^v &=& -\delta(\varphi^*(t))q_t^v dt + v'h(\varphi^*(t))dt \\ q_0^v &=& v's^* = v_1 s_1^* + ... + v_K s_K^*. \end{array}\right. \tag{1.57}$$

A crucial remark is that, up to notation, $\{q_t^v,\ t\geq 0\}$ satisfies a linear equation similar to the one dimensional equation (1.46) (replacing s^* by $v's^*$ and $h(.)$ by $v'h(.)$). Thus we are exactly in the situation of subsection 1.4.1, so that from (1.50) one has expression of $\mathbb{E}((\exp(q_t^v)1_{\{\varphi^*(t)=i\}}) = \mathbb{E}\left(\exp(v'Q_t)1_{\{\varphi^*(t)=i\}}\right)$ for all $v=(v_1,...,v_K)'\in[0,+\infty)^K$.

Step 3 : Achieving computation of (1.55), hence of (1.53). Thanks to Proposition 7 , and by Fubini, for all $I\subset K$ and $i\in E_\Phi$:

$$\begin{aligned} &\int_{x\in\mathcal{D}} e^{v'x}\mathbb{P}\left(A_t^I(x),\ \varphi(0)=i\right)dx \\ =\ &\int_{x=(x_1,...,x_K)\in\mathcal{D}} e^{v'x}\mathbb{P}\left(\bigcap_{j\in I}[\tau^j(x_j)<t],\ \varphi(0)=i\right)dx \\ =\ &\mathbb{E}\left(\int_{x=(x_1,...,x_K)\in\mathcal{D}} e^{v'x}\prod_{j\in I}1_{\{Q_t^j>x_j\}}dx 1_{\{\varphi^*(t)=i\}}\right) \\ =\ &\mathbb{E}\left(\prod_{j\in I}\frac{1}{v_j}\left[e^{v_jQ_t^j}-e^{v_js_j^*}\right]\prod_{j\notin I}\frac{1}{v_j}e^{v_js_j^*}1_{\{\varphi^*(t)=i\}}\right) \\ =\ &\prod_{j=1}^K\frac{1}{v_j}\prod_{j\notin I}e^{v_js_j^*}\times\sum_{A\subset I}\left[\prod_{k\notin A}(-e^{v_ks_k^*})\mathbb{E}\left(\prod_{k\in A}e^{v_kQ_t^k}1_{\{\varphi^*(t)=i\}}\right)\right]. \end{aligned} \tag{1.58}$$

Since, by Step 2, $\mathbb{E}\left(\prod_{k\in A}e^{v_kQ_t^k}1_{\{\varphi^*(t)=i\}}\right)$ has an explicit expression, one plugs (1.58) into the integral with respect to $x \in \mathcal{D}$ of (1.54) to obtain expression of (1.53).

Chapitre 2

Proportional Reinsurance

This chapter is dedicated to a special case of multidimensional risk theory. We will consider here a two dimensional risk process verifying the following conditions :

- Premia are paid at fixed (possibly modulated) rates for each branch.
- There are up to two sources of incoming claims ; each claim from a source is split according to a fixed proportion towards each branch.
- There may be a common interest force for each branch.

Traditionnally, one branch plays the role of an insurance company (the "cedent") which needs to have some of its risky claims covered, and the other one is a reinsurance company, which covers those claims. The situation described here corresponds to *quota share* reinsurance. In the following sections, each aspect (but not all at a time) will be present. The quantities that will be studied in this chapter will be redefined in each part, but will essentially be one of the following

- the exit time out of the first quadrant, i.e. the ruin time of one of the branches,
- the exit time out of the third quadrant, which corresponds to simultaneous ruin of all branches.

In Section 2.1 we consider a model with one source of incoming claims and an interest rate. In Section 2.2 a model with two sources, reinsurance on one source, but no interest rate is considered. At first sight it may seem that Section 2.1 deals with a more general case ; In fact this is not true as the arguments used in Section 2.1 are specific to the model and cannot be used when there is no interest rate. Finally, in Section 2.3 we consider a model with two sources of claims (and no interest rate).

Notation, especially on the model, will be set at the beginning of each section.

2.1 A model with interest rate and one source of claims

This section essentially concerns Section 5.2 of [Rab09], which is somewhat independent from the rest of the article. We consider the following two dimensional risk process :

$$\begin{cases} dR_t^1 &= \delta R_t^1 dt + c_1 dt - \alpha d\mathcal{S}(t) \\ dR_t^2 &= \delta R_t^2 dt + c_2 dt - (1-\alpha) d\mathcal{S}(t) \\ (R_0^1, R_0^2) &= x = (x_1, x_2) \in [0, +\infty)^2 \end{cases} \tag{2.1}$$

which is model (1.51) in the previous chapter with $K = 2$ and no modulating Markov chain. This is for clarity purpose, although the model is a bit more general in [Rab09], and it may certainly be possible in some cases to be even more general than in the paper without adding further technical difficulties in what follows. $\delta > 0$, $c_1 \geq 0$ and $c_2 \geq 0$ are respectively the interest rate and premium rates for each branches. $\alpha \in (0,1)$ is the proportion of incoming claims taken in charge by branch 1, the remaining $1-\alpha$ taken by branch 2. Also, we will suppose that $\{\mathcal{S}(t),\ t \geq 0\}$ is a compound Poisson process, although here again some more general case may be considered.

We recall definition of the definitve ruin thresholds for both lines (see Section 1.4.1) :

$$s_1^* = -c_1/\delta, \quad s_2^* = -c_2/\delta.$$

We study in this Section the exit time out of the first quadrant defined as

$$\mathcal{T}(x) = \mathcal{T}(x_1, x_2) := \inf\{t \geq 0 |\ R_t \notin [0, +\infty)^2\}.$$

This is to be compared with absolute ruin time $\mathcal{T}_*(x)$ defined as exit time out of $\mathcal{D} := [s_1^*, +\infty) \times [s_2^*, +\infty)$ in Section 1.4.2. Studying distribution of $\mathcal{T}(x)$ is more delicate as, as was underlined in the multidmensional case in Section 1.4.2, an important property of $\mathcal{D}$ is that $\mathbb{R}^2 \backslash \mathcal{D}$ is absorbing for $R_t = (R_t^1, R_t^2)'$. This is no longer the case for $\mathbb{R}^2 \backslash [0, +\infty)^2$.

Before turning to the two dimensional problem, a central assumption is that the corresponding one dimensional problem is solved, i.e. that cdf's of ruin times

$$\mathcal{G}^j(t, x_j) := \mathbb{P}(\mathcal{T}_j(x_j) \leq t), \quad \mathcal{T}_j(x_j) := \inf\{t \geq 0 |\ R_t^j < 0\}, \quad j = 1, 2,$$

are available. We recall that the Laplace transforms of $\mathcal{G}^j(t, x_j)$ in the x_j and t variables were computed in [Rab09] when jumps are exponentially distributed (see comment at end of Section 1.4.1), and when interclaims are e.g. Phase type equally distributed. In fact the case when $\{\mathcal{S}(t),\ t \geq 0\}$ is a plain compound Poisson process and claims admit a density is treated in [WWZ05] (see Theorem 3.1 therein), where the authors establish explicit expressions for the density of $\mathcal{T}_j(x_j)$, jointly to surplus before and on ruin of the risk process.

The objective is to get an expression of the cdf of $\mathcal{T}(x)$,

$$\mathcal{G}(t, x) = \mathcal{G}(t, x_1, x_2) = \mathbb{P}(\mathcal{T}(x_1, x_2) \leq t), \quad x = (x_1, x_2) \in \mathcal{D}.$$

A first approach would be to use the Markovian structure of $\{R_t = (R_t^1, R_t^2),\ t \geq 0\}$ and establish a renewal or partial differential equation for $\mathcal{G}(.,.)$ or its Laplace Transform with respect to one or several variables t and x. However, one would have to tackle the difficulty of solving this equation, which may not be easy, one of the reasons being the number of variables involved. We propose an alternative, geometric approach, that turns out well in the present situation, and that yields an explicit expression of $\mathcal{G}(t, x)$ in function of the $\mathcal{G}^j(t, x_j)$'s. Let us first notice that, if one represents trajectories of R_t in $\mathbb{R}^2$, then jumps are all directed along vector $(-\alpha, -1+\alpha)$. We split domain $\mathcal{D}$ into three sets

$$\mathcal{D} = \mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2$$

defined on Figure 2.1, where $\alpha c_2 > (1-\alpha) c_1$ without loss of generality (otherwise, one has just to swap R_t^1 and R_t^2). Lines D_1 and D_2 are defined by the fact that they contain point (s_1^*, s_2^*) and are parrallel to direction of claims, so have the following equation

$$\begin{aligned} D_1 : r_2 &= \frac{1-\alpha}{\alpha} r_1 + \frac{1}{\delta}\left(\frac{1-\alpha}{\alpha} c_1 - c_2\right), \\ D_2 : r_2 &= \frac{1-\alpha}{\alpha} r_1. \end{aligned}$$

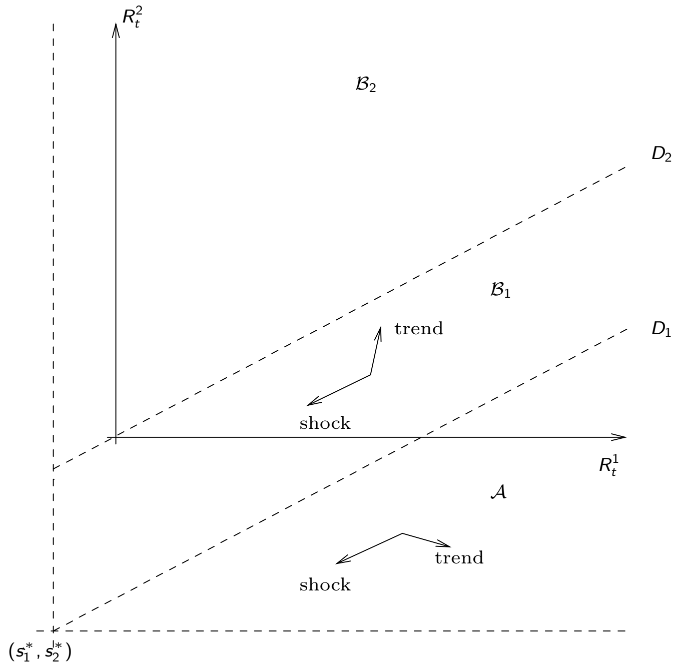

FIGURE 2.1 – Partitioning $\mathcal{D}$ in the case of proportional reinsurance

Let $\mathbf{n} := (1-\alpha, -\alpha)$ be a vector perpendicular to direction of jumps. We define $X_t := <\mathbf{n}, R_t>$. X_t is such that $X_t > 0$ iff $R_t \in \mathcal{A} \cup \mathcal{B}_1$, and $X_t < 0$ iff $R_t \in \mathcal{B}_2$. Furthermore, it is not hard to check that $\{X_t,\ t \geq 0\}$ verifies the linear *deterministic* differential equation

$$\begin{cases} dX_t &= \delta X_t dt + (\mathbf{n}.c)dt \\ X_0 &= \mathbf{n}.x = (1-\alpha)x_1 - \alpha x_2 \end{cases}$$

where $c = (c_1, c_2)'$. Note that the fact $\mathbf{n}.c = (1-\alpha)c_1 - \alpha c_2 < 0$ implies that

$$\begin{array}{ccccc} R_t \in \mathcal{B}_1 \cup \mathcal{B}_2 & \iff & \delta X_t dt + (\mathbf{n}.c) < 0 & \iff & X_t \text{ is decreasing}, \\ R_t \in \mathcal{A} & \iff & \delta X_t dt + (\mathbf{n}.c) > 0 & \iff & X_t \text{ is increasing}. \end{array}$$

(see again Figure 2.1 for the illustration of this property), so that :

- if $(R_0^1, R_0^2) = (x_1, x_2) \in \mathcal{A}$ then $\{R_t = (R_t^1, R_t^2),\ t \geq 0\}$ always remains in $\mathcal{A}$,
- if $(R_0^1, R_0^2) = (x_1, x_2) \in \mathcal{B}_2$ then $\{R_t = (R_t^1, R_t^2),\ t \geq 0\}$ always remains in $\mathcal{B}_2$,
- if on the other hand $(R_0^1, R_0^2) = (x_1, x_2) \in \mathcal{B}_1$, then it will eventually enter $\mathcal{B}_2$, as the trend makes the process move away from D_1.

Sets $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ thus have the following properties : $\mathcal{A}$ and $\mathcal{B}_2$ are *absorbing sets* for the markovian process $\{R_t = (R_t^1, R_t^2),\ t \geq 0\}$, and $\mathcal{B}_1$ is *transient*. This means that if R_t starts in $\mathcal{A}$ (resp. in $\mathcal{B}_2$) then it will leave first quadrant $[0, +\infty)^2$ iff R_t^2 (resp. R_t^1) hits 0. And, since $\{X_t,\ t \geq 0\}$ is deterministic, entrance time $T(x) = T(x_1, x_2)$ of $\{R_t,\ t \geq 0\}$ from $(x_1, x_2) \in \mathcal{B}_1$ into $\mathcal{B}_2$ is deterministic. Easy calculation yields

$$T(x) = T(x_1, x_2) := \frac{1}{\delta} \ln \left(\frac{(1-\alpha)c_1 - \alpha c_2}{(1-\alpha)(\delta x_1 + c_1) - \alpha(\delta x_2 + c_2)} \right). \tag{2.2}$$

All these remarks are summed up in the following result :

Theorem 11. *The expression of* $\mathcal{G}(t, x_1, x_2)$ *is given in the following cases :*

1. *If* $x = (x_1, x_2) \in \mathcal{A}$ *then* $\mathcal{G}(t, x_1, x_2) = \mathcal{G}^2(t, x_2)$.
2. *If* $x = (x_1, x_2) \in \mathcal{B}_2$ *then* $\mathcal{G}(t, x_1, x_2) = \mathcal{G}^1(t, x_1)$.
3. *If* $x = (x_1, x_2) \in \mathcal{B}_1$ *then*

$$\mathcal{G}(t, x_1, x_2) = \begin{cases} \mathcal{G}^2(t, x_2), & t < T(x) \\ \mathcal{G}^2(T(x), x_2) + \mathcal{G}^1(t, x_1) - \mathcal{G}^1(T(x), x_1), & t \geq T(x), \end{cases}$$

where $T(x)$ *is entrance time of* $\{R_t,\ t \geq 0\}$ *into* $\mathcal{B}_2$*, given by (2.2).*

2.2 A model with two sources of claim.

This section concerns [BCR11]. We consider the following risk process $\{Y_t = (Y_t^1, Y_t^2),\ t \geq 0\}$ that satisfies

$$\begin{cases} dY_t^1 = p_1\, dt - a\, dL_t - dS_t, \\ dY_t^2 = p_2\, dt - (1-a)\, dL_t, \\ (Y_0^1, Y_0^2) = (y_1, y_2), \end{cases} \tag{2.3}$$

p_1 and p_2 are the premium rates. Claim sources are represented by $\{L_t,\ t \geq 0\}$ and $\{S_t,\ t \geq 0\}$, two independent compound Poisson processes with general jump (claim) distribution. $a \in (0, 1)$ is the proportion of claims from L_t taken in charge by Y_t^1, and $1 - a$ is taken in charge by Y_t^2. Since reinsurance is made on one type of claim, we will call Y^2 the *reinsurer* and Y^1 the *cedent*, so that claims from process S_t are entirely covered by the cedent. This can also of course model a scenario where Y_t^1 and Y_t^2 are two branches of one insurance company, and branch 2 covers a part of one type of claims.

We aim at determining the Laplace transform $\mathbb{E}_{(y_1,y_2)}(e^{-\beta\tau})$ of the exit time out of the first quadrant defined as in the previous section

$$\tau = \inf\{t \geq 0 | Y_t \notin [0, +\infty)^2\} = \inf\{t \geq 0 | \min(Y_t^1, Y_t^2) < 0\} = \min(\tau_1, \tau_2). \tag{2.4}$$

where $\tau_i = \inf\{t \geq 0 | Y_t^i < 0\}$ is ruin time of process Y_t^i, $i = 1, 2$. Note that this generalizes [APP08a] and [APP08b], where the authors consider a model where reinsurance is also made on one type of claims, but where jumps are exponentially distributed, and there is only one source of claims.

2.2.1 Prior geometrical remarks : exhibiting an absorbing set $\mathcal{A}^- \subset \mathbb{R}^2$.

As in Section 2.1, we will resort to geometric considerations in order to tackle this problem. We will suppose that the following inequality holds

$$\frac{p_2}{p_1} > \frac{1-a}{a}. \tag{2.5}$$

Let us try to interpret Condition (2.5). Let $\theta_1 > 0$ and $\theta_2 > 0$ be safety loadings of cedent with respect to claims L_t and S_t. Likewise, let $\theta_3 > 0$ be the safety loading for reinsurer with respect to claims L_t (the only ones he reinsures). Then premium rates may be written as

$$\begin{aligned} p_1 &= (1+\theta_1)a\mathbb{E}[L_1] + (1+\theta_2)\mathbb{E}[S_1] \\ p_2 &= (1+\theta_3)(1-a)\mathbb{E}[L_1]. \end{aligned}$$

A healthy assumption is that $\theta_3 > \theta_1$, i.e. that the second line has a higher safety loading. Indeed, Y_t^2 potentially takes a risk by reinsuring Y_t^1, so it seems legitimate that, by compensation, it has a high security loading. In that case (2.5) reads

$$\frac{\mathbb{E}[L_1]}{\mathbb{E}[S_1]} > \frac{1+\theta_2}{a(\theta_3-\theta_1)},$$

which roughly says that claims issued from L_t are in average larger and more frequent (i.e., riskier) than those from S_t. Therefore, it sounds logical to especially reinsure those riskier claims.

As in Section 2.1, we turn to geometrical considerations. We define line $\Delta \subset \mathbb{R}^2$ of equation

$$\Delta : \ y = \frac{1-a}{a}x,$$

so that claims occurring according to L_t are thus parallel to Δ. Next, sets $\mathcal{A}^+$ and $\mathcal{A}^- \subset \mathbb{R}^2$ are defined by

$$\mathcal{A}^+ := \{x \in \mathbb{R}^2 | \ < x, \mathbf{v} > > 0\}, \text{ and } \mathcal{A}^- := \{x \in \mathbb{R}^2 | \ < x, \mathbf{v} > < 0\}, \quad \text{where } \mathbf{v} = (1-a, -a)'.$$

Vector $\mathbf{v}$ is orthogonal to Δ (and thus to direction of claims). Similarly to Section 2.1, we define X_t by $X_t := < \mathbf{v}, (Y_t^1, Y_t^2) >$. X_t may be seen as an (algebraic) distance between $Y_t \in \mathbb{R}^2$ and Δ, and is such that

$$X_t > 0 \iff (Y_t^1, Y_t^2) \in \mathcal{A}^+, \quad X_t < 0 \iff (Y_t^1, Y_t^2) \in \mathcal{A}^-.$$

Last, a central remark is that Condition 2.5 implies that $\mathcal{A}^-$ is an *absorbing set*. This is illustrated in Figure 2.2 This absorbing property can be easily seen thanks to the fact that X_t verifies

$$\begin{cases} dX_t = [(1-a)p_1 - ap_2]\, dt - (1-a)\, dS_t, \\ X_0 = < \mathbf{v}, (y_1, y_2) > = (1-a)y_1 - ay_2 \end{cases} \tag{2.6}$$

with $(1-a)p_1 - ap_2 < 0$, so is decreasing. Note that, contrarily to [APP08a], [APP08b] and Section 2.1, X_t is no longer deterministic.

2.2.2 Strategy for determining $\mathbb{E}_{(y_1,y_2)}(e^{-\beta\tau})$.

In function of where Y_t lies at $t = 0$, two outcomes are possible in order to determine Laplace transform of τ. Either $Y_0 = (y^1, y^2) \in \mathcal{A}^-$: Then, Y_t remains in this absorbing set, and $\tau = \tau_1$ corresponds to ruin time of Y_t^1. Or $Y_0 = (y^1, y^2) \in \mathcal{A}^+$. In that case, there is a competition between hitting time τ_X of line Δ and ruin time τ_2 of Y_t^2 :

- if Y_t^2 is ruined before process Y_t hits Δ then exit out of the first quadrant is at time $\tau = \tau_2$,
- if on the other hand Y_t hits Δ before Y_t^2 hits 0 then Y_t will then remain in $\mathcal{A}^-$ and exit out of the first quadrant (if it happens) will correspond to ruin of Y_t^1.

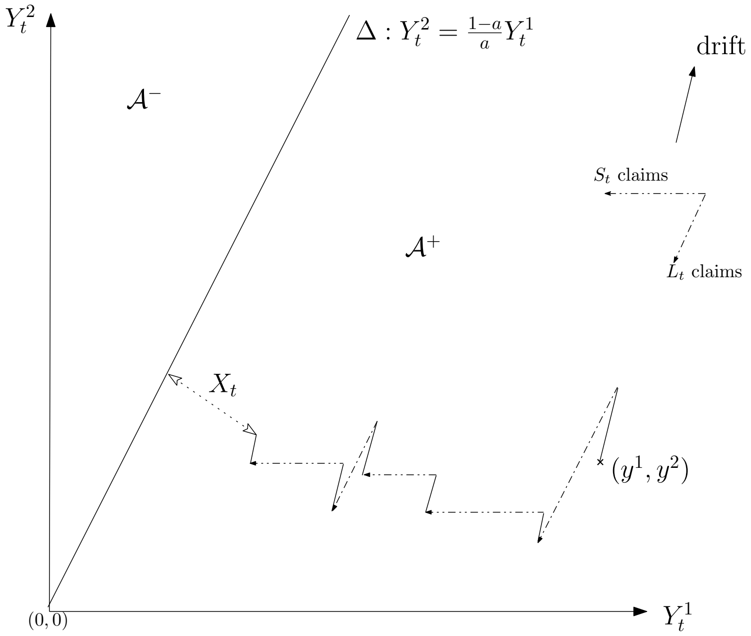

FIGURE 2.2 – A sample path of $\{(Y_t^1, Y_t^2),\ t \geq 0\}$.

In the case where $(y^1, y^2) \in \mathcal{A}^-$, $\tau = \tau_1$ corresponds to ruin time of a classical Cramer model with Poisson arrival and claims admitting a density, or more broadly to a special case of spectrally negative Lévy process, of which Laplace transform can be found in the literature (see e.g. Lemma 3.1 p.126 of [AA10], or Theorem 8.1 (ii) of [Kyp06] for a representation that involves so-called scale functions) ; alternatively, its density may also be found in [DW05]. We will then suppose that

$$(y^1, y^2) \in \mathcal{A}^+$$

which is the most delicate case. The ingredients for determining Laplace transform of τ are the following : We will require

- expression of the distribution of ruin of X_t and deficit at ruin, (τ_X, X_{τ_X}),
- expression of the distribution of Y_t^2 conditioned to stay positive.

Another important remark is that, as observed from (2.3) and (2.6), $\{Y_t^2,\ t \geq 0\}$ and $\{X_t,\ t \geq 0\}$ respectively depend on $\{L_t,\ t \geq 0\}$ and $\{S_t,\ t \geq 0\}$ so are *independent*.

Step 1 : Expression of distribution of ruin of X_t and deficit at ruin. We rewrite (2.6) as

$$dX_t = -c\,dt - dS_t^a, \quad X_0 = x, \tag{2.7}$$

with $c = ap_2 - (1-a)p_1 > 0$ and $S_t^a = (1-a)S_t$. Since, from (2.7), X_t is a decreasing process with negative jumps, τ_X admits a mass at x/c and X_{τ_X} at 0. By using a renewal equation verified by the Laplace transform of (τ_X, X_{τ_X}), then adequately inverting it, one gets an expression of quantity

$$\mathbb{P}_x(\tau_X \in dt, X_{\tau_X} \in dz) = k_X \delta_{x/c}(dt) + h_C(t|x)dt.\delta_0(dz) + h_J(z, t|x)dtdz \tag{2.8}$$

for some constant $k_X > 0$ and densities $h_C(.|x)$ and $h_J(., .|x)$. These densities feature multiple convolutions of densities of claim sizes, which is not very convenient from a practical point of

view ; however one may be relieved to know that, when these claims admit for instance Erlang distributions then closed form expressions of these densities are available.
Step 2 : distribution of Y_t^2 conditioned to stay positive. We write evolution of Y_t^2 in (2.3) more simply

$$dY_t^2 = p_2\, dt - dL_t^a \tag{2.9}$$

where we here specify λ_L and $f_{a,L}(.)$ as respectively intensity and density of jumps in L_t^a. The starting point is formula from e.g. Lemma 1 of [Ber96]

$$\int_{t=0}^{\infty} e^{-\beta t}\mathbb{P}_{y_2}\left(\inf_{s\le t} Y_s^2 > 0,\, Y_t^2 \in du\right)\, dt = \left[e^{-\rho u}W^{(\beta)}(y_2) - 1_{\{y_2\ge u\}}W^{(\beta)}(y_2-u)\right]\, du, \tag{2.10}$$

where $W^{(\beta)}(.)$ is the scale function defined via its Laplace transform

$$\int_0^\infty e^{-sx}W^{(\beta)}(x)\, dx = \frac{1}{p_2 s - (\lambda_L+\beta) + \lambda_L \widetilde{f}_{a,L}(s)}, \qquad s > \rho, \tag{2.11}$$

where $\widetilde{f}_{a,L}(s) = \int_0^\infty e^{-sx}f_{a,L}(x)\, dx$, and ρ appearing in (2.10) is the unique non-negative root to the equation (in ξ)

$$p_2\xi - (\lambda_L+\beta) + \lambda_L\widetilde{f}_{a,L}(\xi) = 0.$$

It is not always easy to get explicit expressions for $W^{(\beta)}(x)$. However, it turns out that computation of the lefthandside of (2.10) appears in [CL10] in terms of convolutions of $f_{a,L}(.)$. As in for the expressions of $h_C(.|x)$ and $h_J(.,.|x)$ in Step 1, those convolutions may be practically complex to implement numerically. However again, one can be comforted by the fact that those scale functions have closed expressions in many cases, e.g. when claims are Erlang or Phase type distributed (see multiple examples in [HK11]). All in all, one proves that one has an expression of the form

$$\mathbb{P}_{y_2}\left(\inf_{s\le t} Y_s^2 > 0,\, Y_t^2 \in du\right) = k_Y\delta_{y_2+p_2 t}(du) + \zeta(y_2,t,u)\, du \tag{2.12}$$

for some mass $k_Y > 0$ at $u = y_2 + p_2 t$ and a density $\zeta(y_2, t, .)$.
Step 3 : Laplace transform of τ starting from $(y_1, y_2) \in \mathcal{A}^+$. According to whether Y_t^2 hit 0 before Y_t hit Δ or not, we decompose the Laplace transform in

$$\mathbb{E}_{(y_1,y_2)}\left[e^{-\beta\tau}1_{\{\tau<\infty\}}\right] = \mathbb{E}_{(y_1,y_2)}\left[e^{-\beta\tau_2}1_{\{\tau_X>\tau_2\}}\right] + \mathbb{E}_{(y_1,y_2)}\left[e^{-\beta\tau_1}1_{\{\tau_X\le\tau_2,\tau_1<\infty\}}\right]. \tag{2.13}$$

Since τ_X and τ_2 are independent, first term on the righthandside of (2.13) is easily dealt with as

$$\mathbb{E}_{(y_1,y_2)}\left[e^{-\beta\tau_2}1_{\{\tau_X>\tau_2\}}\right] = \int_{u=0}^\infty\int_{t=u}^\infty e^{-\beta t}\mathbb{P}_x(\tau_X\in dt)\mathbb{P}_{y_2}(\tau_2\in du), \tag{2.14}$$

where $x = (1-a)y_1 - ay_2$. $\mathbb{P}_x(\tau_X \in dt)$ is given by (2.8), and $\mathbb{P}_{y_2}(\tau_2 \in du)$ is available e.g. in [DW05]. Let us turn to second term on the righthandside of (2.13). Noticing that, on crossing Δ, position of cedent is

$$Y_{\tau_X}^1 = \frac{a}{1-a}Y_{\tau_X}^2 + \frac{1}{1-a}X_{\tau_X},$$

one uses (2.8) and (2.12) as well as the Markov property and writes

$$\begin{aligned}
\mathbb{E}_{(y_1,y_2)}\left[e^{-\beta\tau_1}1_{\{\tau_X\le\tau_2,\tau_1<\infty\}}\right] &= \int_{t=0}^{\infty}\int_{z=-\infty}^{0}\int_{u=0}^{\infty}\mathbb{E}_{\frac{a}{1-a}u+\frac{1}{1-a}z}\left[e^{-\beta\tau_1}1_{\{\tau_1<+\infty\}}\right] \\
&\quad .\mathbb{P}_{y_2}\left(\inf_{s\le t} Y_s^2>0,\, Y_t^2\in du\right)\mathbb{P}_x(\tau_X\in dt, X_{\tau_X}\in dz) \\
&= \int_{t=0}^{\infty}\int_{z=-\infty}^{0}\int_{u=0}^{\infty}\mathbb{E}_{\frac{a}{1-a}u+\frac{1}{1-a}z}\left[e^{-\beta\tau_1}1_{\{\tau_1<+\infty\}}\right] \\
&\quad .\left[k_Y\delta_{y_2+p_2t}(du)+\zeta(y_2,t,u)\,du\right] \\
&\quad .\left[k_X\delta_{x/c}(dt)+h_C(t|x)dt.\delta_0(dz)+h_J(z,t|x)dtdz\right]
\end{aligned}$$

where we recall that $\mathbb{E}_{\frac{a}{1-a}u+\frac{1}{1-a}z}\left[e^{-\beta\tau_1}1_{\{\tau_1<+\infty\}}\right]$ is available from [AA10], or by integrating density of τ_1 in [DW05].

2.3 Another model with no absorbing set.

This section concerns [Rab12]. The two dimensional risk process studied here is denoted by $\{(X_t^1, X_t^2),\ t\ge 0\}$ and satisfies

$$\begin{cases} X_t^1 &= x_1+\int_0^t p_1(J(s))ds - aS_t - bB_t, \\ X_t^2 &= x_2+\int_0^t p_2(J(s))ds-(1-a)S_t-(1-b)B_t, \end{cases} \tag{2.15}$$

where $\{J(s),\ s\ge 0\}$ is an irreducible stationary finite Markov chain of generator matrix $Q=(q_{ij})_{i,j=1,\dots,K}$ and distribution the row vector $\pi=(\pi_i)_{i=1,\dots,K}$. $\{S_t,\ t\ge 0\}$ is a Markov additive process, i.e. a pure jump process of which the jumps occur at transition times of the Markov chain, of which jumps are of size distributed as U_{ij}^n at time T_n such that $J(s)$ jumps from state i to j a T_n. We suppose that the $(U_{ij}^n)_{n\in\mathbb{N}}$ are independent, light tailed, with moment generating function

$$\varphi_{ij}(x)=\mathbb{E}(e^{xU_{ij}}).$$

$\{B_t,\ t\ge 0\}$ is an independent fractional Brownian motion of Hurst parameter $H\in[1/2,1)$, and a and b lie in $(0,1)$.

The model is motivated by the following remarks :

- $\{S_t,\ t\ge 0\}$ models occurrence of claims, as in Sections 2.1 and 2.2. a (resp. $1-a$) is the proportion of those claims taken in charge by branch X_t^1 (resp. X_t^2).
- even though it is a continuous process, $\{B_t,\ t\ge 0\}$ can be seen as an approximation of a risk process where claims are strongly dependent (see [Mic98] and [Bur00] for the approximation procedure), which corresponds to the "diffusion approximation" when claims are independent (see Chapter V, Section 5 of [AA10]) or the equivalent "heavy traffic" approximation in queueing theory (see e.g. Chapter 6 Section 4 of [CY01]). b is the proportion of these claims taken in charge by X_t^1, and $1-b$ by X_t^2.

Since process $\{B_t,\ t\ge 0\}$ is an approximation of a risk process (which is why it is not increasing), b also affects its premium rate and is standardly referred to as the *risk exposure*, see Introduction of [TM03]. Since $\{B_t,\ t\ge 0\}$ is continuous and obtained by dilating time and shrinking claim sizes in the risk process, we may consider it as risk process with *small claims*, as opposed to $\{S_t,\ t \geq 0\}$ that models *large claims*. For example, $\{B_t,\ t \geq 0\}$ models minor (but nonetheless serious !) events that happen very frequently such as car accidents, personal injuries, small fire, etc., whereas $\{S_t,\ t \geq 0\}$ may model more dramatic and rarer events such as flash floods, major earthquakes and so on.

In the sequel we let the $K \times K$ matrices of moment generating functions and mean jump sizes

$$\varphi(x) := (\varphi_{ij}(x))_{i,j=1,\dots,K}, \quad M = (m_{ij})_{i,j=1,\dots,K} := (\mathbb{E}(U_{ij}))_{i,j=1,\dots,K} = \varphi'(0).$$

We are interested in this section in the exit times out of the first quadrant $[0,+\infty)^2$ and entrance time into third quadrant $(-\infty,0]^2$:

$$\begin{aligned} \tau_{\text{or}} &:= \inf\{t \geq 0 |\ X_t^1 < 0 \text{ or } X_t^2 < 0\}, \\ \tau_{\text{sim}} &:= \inf\{t \geq 0 |\ X_t^1 < 0 \text{ and } X_t^2 < 0\}, \end{aligned}$$

which respectively corresponds to ruin of one or both branches, see Figure 2.3. Mainly because

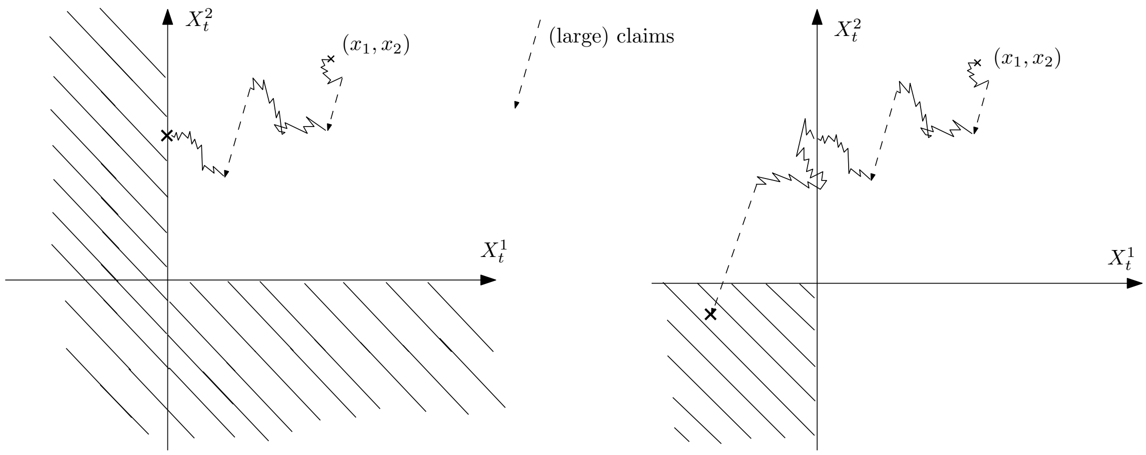

FIGURE 2.3 – Ruin of one or both branches : a reinsurance problem.

of the presence of the fractional Brownian motion, $\{X_t = (X_t^1, X_t^2),\ t \geq 0\}$ does not have nice properties such as being a Markov process, a martingale etc., so that it appears very difficult to derive the exact distribution of τ_{or} or τ_{sim} using standard tools and techniques. Note that, even in the one dimensional case, few is known of the first hitting time of 0 of a drifted fractional Brownian motion (a bound on its Laplace transform is available in [DN08]). Not only that, but the mix of Markov additive process and $\{B_t,\ t \geq 0\}$ makes the problem even less tractable. It may however be interesting to know that, in the case of a pure N dimensional Gaussian process, asymptotics results already exist concerning τ_{sim}, as seen in [DKMR10], or in discrete time setting as in [Has05]. We will therefore consider the two following probabilities of eventual ruin starting from $(X_0^1, X_0^2) = (x_1, x_2)$

$$\psi_{\text{or}}(x_1,x_2) := \mathbb{P}\left(\tau_{\text{or}} < +\infty |\ (X_0^1, X_0^2) = (x_1, x_2)\right), \ \psi_{\text{sim}} := \mathbb{P}\left(\tau_{\text{sim}} < +\infty |\ (X_0^1, X_0^2) = (x_1, x_2)\right).$$

Since exact expressions for these quantities look difficult to obtain, we are looking for asymptotics of the form

$$\frac{1}{x_1^{2-2H}} \ln \psi_{\text{or}}(x_1, x_2), \quad \frac{1}{x_1^{2-2H}} \ln \psi_{\text{and}}(x_1, x_2) \sim -C^*, \quad x_1 \to +\infty, \quad x_2/x_1 = \beta,$$

where $C^* > 0$ does depend on H, β, as well as on other parameters of the model. In the Lévy case, these kind of asymptotics are investigated in [APP08b]. In fact, $H = 1/2$, i.e. when B_t is a Lévy process (a brownian motion), is a special case that is dealt with in [Rab12] but not mentioned in the present document. In upcoming Subsection 2.3.1, we will study τ_{or}. In Subsection 2.3.2, we will study τ_{sim} in the case of no Markov modulation, i.e. when S_t is a plain compound Poisson process, and in the particular case when Hurst parameter verifies $H \in (5/6, 1)$. Although we believe that considering a Markov additive process may be possible but may just only add technicalities, condition on the Hurst parameter is a real technical constraint, and, surprisingly, proofs do not seem to work if $H \in [1/2, 5/6]$ although we suspect that the same results hold in that case.

The central steps that we will use for proving asymptotics for $\psi_{\text{or}}(x_1, x_2)$ and $\psi_{\text{sim}}(x_1, x_2)$ are the following. We will first reduce the two dimensional problem to a 1 dimensional one, as is implicitly done in [APP08b] as well as in Sections 2.1 and 2.2. Then we will use a result by Duffield and O'Connell [DO95] which may be summed up in the following way (tailored to our need) :

Theorem 12 (Duffield, O'Connell (95)). *Let $\{W_t,\ t \geq 0\}$ be a real valued process. Let us suppose that the following assumptions hold :*

(i) the cumulant generating function defined as

$$\lambda(\theta) := \lim_{t \to +\infty} \frac{1}{t^{2-2H}} \ln \mathbb{E} e^{\theta t^{1-2H} W_t}$$

exists in $[-\infty, +\infty]$ and verifies $\lambda(\theta) < 0$ for $\theta > 0$ close to 0,

(ii) $W_n^* := \sup_{0 \leq r < 1} W_{n+r}$ verifies $\limsup_{n \to +\infty} \frac{1}{n^{2-2H}} \ln \mathbb{E} e^{\theta n^{1-2H}(W_n^* - W_n)} = 0$ for all $\theta > 0$.

Then, if the Fenchel-Legendre function

$$\lambda^*(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \lambda(\theta)\}$$

is continuous on $x \geq 0$, one has that

$$\lim_{b \to +\infty} \frac{1}{b^{2-2H}} \ln \mathbb{P}\left(\sup_{t \geq 0} W_t > b\right) = -\inf_{z > 0} z^{-(2-2H)} \lambda^*(z). \tag{2.16}$$

The main technical issues will essentially

- to be able to identify $\{W_t,\ t \geq 0\}$ in each case τ_{or} and τ_{sim},
- then to determine closed expressions of the corresponding cumulant generating function $\lambda(\theta)$ and its associated Fenchel-Legendre tranform $\lambda^*(x)$.

Condition on W_n^* will of course also be checked, although it turns out this is not the most demanding part.

Before giving main results, we set $\bar{p_1} = \sum_{i=1}^K p_1(i)\pi_i$, $\bar{p_2} = \sum_{i=1}^K p_2(i)\pi_i$ and introduce the safety loadings, that we suppose positive,

$$\rho_1 := \frac{\bar{p_1}}{a\mathbb{E}(S_1)} - 1 > 0, \quad \rho_2 := \frac{\bar{p_2}}{(1-a)\mathbb{E}(S_1)} - 1 > 0 \tag{2.17}$$

where $\mathbb{E}(S_1) = \sum_{i,j=1,\ldots,K} \pi_i q_{ij} m_{ij}$. For presentation purpose, we will suppose that $\frac{1-b}{b} < \frac{(1-a)\rho_2}{a\rho_1}$, although similar results hold when inequality is reversed. We also let

$$\theta_\beta := 2\frac{\bar{p_1} - \bar{p_2}/\beta + (-a + (1-a)/\beta)\mathbb{E}(S_1)}{b^2 - (1-b)^2/\beta^2}, \tag{2.18}$$

2.3.1 Asymptotics for $\psi_{\mathbf{or}}(x_1, x_2)$.

The result obtained in the case $H \in (1/2, 1)$ is the following :

Theorem 13. *Letting*

$$\begin{aligned} \lambda_1(\theta) &:= \theta\left[-\bar{p_1} + a\mathbb{E}(S_1)\right] + b^2\frac{\theta^2}{2} \\ \lambda_2(\theta) &:= \theta\frac{-\bar{p_2} + (1-a)\mathbb{E}(S_1)}{\beta} + \frac{(1-b)^2}{\beta^2}\frac{\theta^2}{2}, \end{aligned} \tag{2.19}$$

$$\lambda(\theta) = \max\left(\lambda_1(\theta), \lambda_2(\theta)\right).$$

Then the following asymptotic holds

$$\lim_{x_1 \to +\infty,\ x_2/x_1 = \beta} \frac{1}{x_1^{2-2H}} \ln \psi_{\text{or}}(x_1, x_2) = -\inf_{z>0} z^{-(2-2H)}\lambda^*(z) := -C^*_{\text{or}}(H, \beta) \tag{2.20}$$

where $\lambda^*(.)$ is given by Figure 2.4.

Contrarily to Sections 2.1 and 2.2, there is no absorbing set in $\mathbb{R}^2$ for process $\{(X^1_t, X^2_t),\ t \geq 0\}$ because of the fractional brownian motion. However, there seems to be three characteristic regions (cones) in $\mathbb{R}^2$ as represented in Figure 2.4 where behavior of the ruin probability is different when the initial reserves tend to infinity along a direction contained in those cones.

The main steps for proving Theorem 13 are given in the following.

Step 1 : reduction to one dimensional problem. We have the following equivalences :

$$\begin{aligned} \tau_{\text{or}} < +\infty &\iff \inf_{t\geq 0} X^1_t < 0 \quad \text{or} \quad \inf_{t\geq 0} X^2_t < 0 \quad \iff \quad \inf_{t\geq 0} X^1_t < 0 \quad \text{or} \quad \inf_{t\geq 0} \frac{1}{\beta}X^2_t < 0 \\ &\iff \sup_{t\geq 0} -x_1 - \int_0^t p_1(J(s))ds + aS_t + bB_t > 0 \\ &\qquad \text{or} \quad \sup_{t\geq 0} \frac{1}{\beta}\left[-x_2 - \int_0^t p_2(J(s))ds + (1-a)S_t + (1-b)B_t\right] > 0 \\ &\iff \sup_{t\geq 0} \max\Bigg(-x_1 - \int_0^t p_1(J(s))ds + aS_t + bB_t\ , \\ &\frac{1}{\beta}\left[-x_2 - \int_0^t p_2(J(s))ds + (1-a)S_t + (1-b)B_t\right]\Bigg) > 0. \end{aligned} \tag{2.21}$$

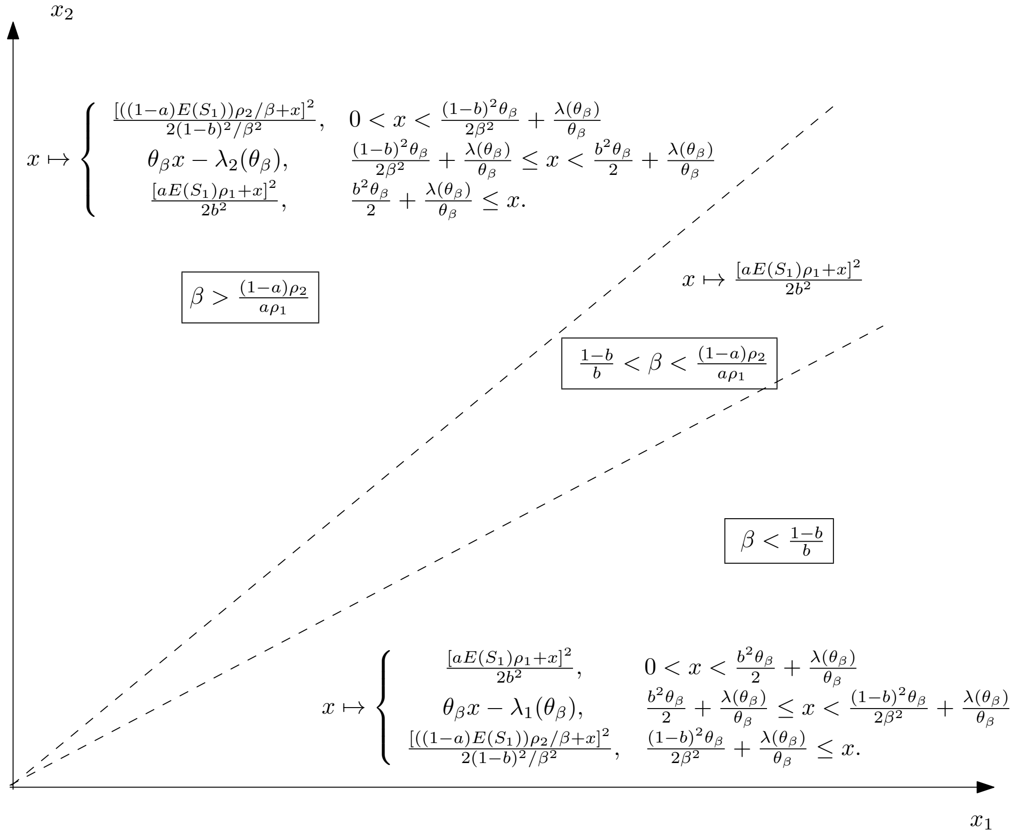

FIGURE 2.4 – Expressions of $\lambda^*(.)$ in the "or" case.

As a consequence, we set

$$A_t^1 := -\int_0^t p_1(J(s))ds + aS_t + bB_t, \tag{2.22}$$

$$A_t^2 := \frac{1}{\beta}\left[-\int_0^t p_2(J(s))ds + (1-a)S_t + (1-b)B_t\right]. \tag{2.23}$$

$$Z_t^{\text{or}} := \max\left(A_t^1, A_t^2\right),$$

so that from (2.21) we have, along $x_2/x_1 = \beta$,

$$\psi_{\text{or}}(x_1, x_2) = \psi_{\text{or}}(x_1, \beta x_1) = \mathbb{P}\left(\sup_{t\ge 0} \max\left(A_t^1, A_t^2\right) > x_1\right) = \mathbb{P}\left(\sup_{t\ge 0} Z_t^{\text{or}} > x_1\right). \tag{2.24}$$

The objective is then to use Theorem 12 with $W_t := Z_t^{\text{or}}$ so defined.

Step 2 : Determining $\lambda(\theta)$ in the "or" case. one writes

$$\mathbb{E}\left(e^{\theta t^{1-2H} Z_t^{\text{or}}}\right) = \mathbb{E}\left(e^{\theta t^{1-2H} A_t^1} 1_{\{A_t^2 \le A_t^1\}}\right) + \mathbb{E}\left(e^{\theta t^{1-2H} A_t^2} 1_{\{A_t^2 > A_t^1\}}\right) := P_1(t) + P_2(t).$$

The starting point is the following set of inequalities :

$$\mathbb{E}\left(e^{\theta t^{1-2H}A_t^1}\right) \geq P_1(t) \geq \mathbb{E}\left(e^{\theta t^{1-2H}A_t^1}\right) - \mathbb{E}\left(e^{\theta t^{1-2H}A_t^2}\right). \tag{2.25}$$

$$\mathbb{E}\left(e^{\theta t^{1-2H}A_t^2}\right) \geq P_2(t) \geq \mathbb{E}\left(e^{\theta t^{1-2H}A_t^2}\right) - \mathbb{E}\left(e^{\theta t^{1-2H}A_t^1}\right), \tag{2.26}$$

which shows intuitively that

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}e^{\theta t^{1-2H}Z_t^{\rm or}} &= \lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}\left[P_1(t)+P_2(t)\right]\\ &= \begin{cases}\lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}\left(e^{\theta t^{1-2H}A_t^1}\right),\\ \text{or } \lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}\left(e^{\theta t^{1-2H}A_t^2}\right),\end{cases}\end{aligned} \tag{2.27}$$

according to whichever is predominant. One then proves rigorously that equality, which then leads to determining an expression for

$$\lambda_i(\theta) := \lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}\left(e^{\theta t^{1-2H}A_t^i}\right), \quad i=1,2. \tag{2.28}$$

Considering for example $\lambda_1(\theta)$, using definition (2.22), as well as the fact that $\{(J(t),S_t),\ t\geq 0\}$ is independent from $\{B_t,\ t\geq 0\}$,

$$\mathbb{E}\left(e^{\theta t^{1-2H}A_t^1}\right) = \mathbb{E}\left(e^{\theta t^{1-2H}\left[-\int_0^t p_1(J(s))ds+aS_t\right]}\right).\mathbb{E}\left(e^{\theta t^{1-2H}bB_t}\right) \tag{2.29}$$

with $\mathbb{E}\left(e^{\theta t^{1-2H}bB_t}\right) = e^{b^2\theta^2 t^{2-2H}/2}$. The trickiest part lies in finding

$$\lim_{t\to+\infty}\frac{1}{t^{2-2H}}\ln\mathbb{E}\left(e^{\theta t^{1-2H}\left[-\int_0^t p_1(J(s))ds+aS_t\right]}\right).$$

which is done through a famous martingale result by Asmussen and Kella on Markov additive processes (more precisely, Lemma 2.1 of [AK00]) as well as technicalities not detailed here. All in all, one proves that $\lambda_1(\theta)$ defined by (2.28) has Expression (2.19), and that $\lambda(\theta)$ is from (2.27), given by $\lambda(\theta)=\max(\lambda_1(\theta),\lambda_2(\theta))$.

Step 3 : Determining $\lambda^*(x)$ and checking all conditions in Theorem 12. Without giving too many details, it turns out that $\lambda(\theta)$ determined previously is defined piecewise, on different intervals, and is either linear or quadratic on each of these intervals, which simplifies a bit computation of $\lambda^*(x)$. As to the technical condition in Theorem 12 concerning W_n^*, one essentially uses simple properties on the supremum of continuous gaussian processes on a finite interval, of which tail is fast decreasing.

2.3.2 Asymptotics for $\psi_{\text{sim}}(x_1,x_2)$.

As mentioned before, we suppose here that we are in the nonmodulated case, which means that there is no external Markov chain $\{J(t),\ t\geq 0\}$, that $p_i(.)=p_i$, $i=1,2$, is a constant, and that process $\{S_t,\ t\geq 0\}$ is a plain Poisson process of intensity $\lambda>0$ of which jumps have an expectation m.

The result obtained in the case $H\in(5/6,1)$ is the following :

Theorem 14. *In the case $H \in (5/6, 1)$, we have*

$$\lim_{x_1 \to +\infty,\ x_2/x_1 = \beta} \frac{1}{x_1^{2-2H}} \ln \psi_{\text{sim}}(x_1, x_2) = - \inf_{z>0} z^{-(2-2H)} \lambda^*(z) := -C^*_{\text{sim}}(H, \beta) \tag{2.30}$$

where the Fenchel Legendre transform $\lambda^*(.)$ is given by Figure 2.5.

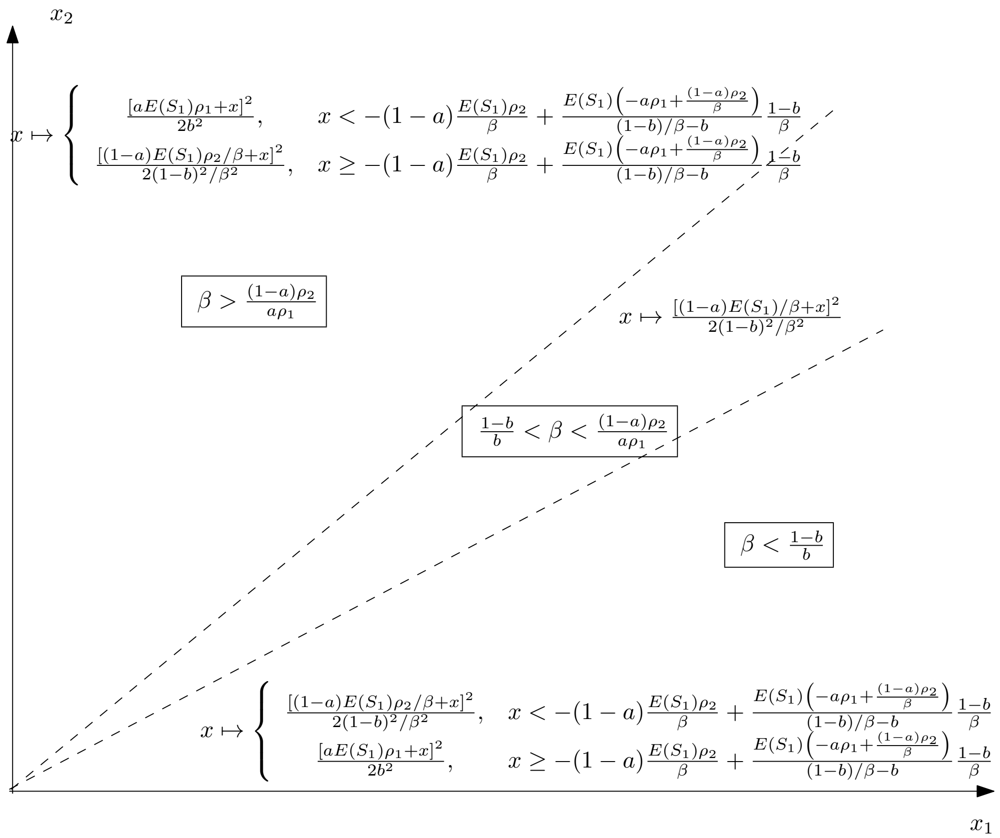

FIGURE 2.5 – Expressions of $\lambda^*(.)$ in the "sim" case.

As in the previous subsection, we provide a sketch of proof :
Step 1 : reduction to one dimensional problem. A similar argument as in the "or" problem yields, with same notation (2.22) and (2.23), and as in (2.24) :

$$\psi_{\text{sim}}(x_1, x_2) = \mathbb{P}\left(\sup_{t \geq 0} Z_t^{\text{sim}} > x_1 \right)$$

with $Z_t^{\text{sim}} := \min(A_t^1, A_t^2)$.
Step 2 : Determining $\lambda(\theta)$ in the "sim" case. One writes this time

$$\mathbb{E}\left(e^{\theta t^{1-2H} Z_t^{\text{sim}}}\right) = \mathbb{E}\left(e^{\theta t^{1-2H} A_t^2} 1_{\{A_t^2 \leq A_t^1\}}\right) + \mathbb{E}\left(e^{\theta t^{1-2H} A_t^1} 1_{\{A_t^2 > A_t^1\}}\right) := Q_1(t) + Q_2(t).$$

The main difference with the "or" case is that a set of inequalities of the kind (2.25) or (2.26) do not hold with $Q_1(t)$ and $Q_2(t)$. However, one proves that if drift of $A_t^1 - A_t^2$ is positive (resp. negative) then $Q_2(t) = o(Q_1(t))$ and $Q_1(t) \sim \mathbb{E}\left(e^{\theta t^{1-2H} A_t^2}\right)$ (resp. $Q_1(t) = o(Q_2(t))$ and $Q_2(t) \sim \mathbb{E}\left(e^{\theta t^{1-2H} A_t^1}\right)$) as $t \to \infty$. This is at this stage that Condition $H \in (5/6, 1)$ plays a part. Since sign of drift of $A_t^1 - A_t^2$ depends on whether β lies in different intervals, one thus gets a different expression for $\lambda(\theta)$ in function of in which of these intervals β lies.

Step 3 : Determining $\lambda^*(x)$ and checking all conditions in Theorem 12. As in the "or" case, the fact that $\lambda^*(x)$ has different expressions on separate intervals comes from definition of $\lambda(\theta)$. And, the technical condition in Theorem 12 is verified similarly.

Chapitre 3

Other topics

We present here published works that are not directly related to the previous chapters. They are however "Risk" or "Queueing" theory flavored, as their applications or tools concern one (or both) of these two fields. In Section 3.1 we study an open-loop optimization problem in a simple queueing network. In Section 3.2 we see how the embedding method can, as in Section 1.4, be adapted to risk process perturbed by a Brownian motion. In Section 3.3, we are interested in a particular risk process with diffusion and we try to find ways of giving a representation more appealing from a computational point of view. Last, in Section 3.4, we see how some results on one sided jump Lévy processes can be applied in a Reliability setting.

3.1 Queues and optimization

This part concerns [GR07]. Let $a = \{a_n,\ n \geq 1\}$ be a stationary sequence of Bernoulli distributed random variables. Let $\{N_t(a),\ t \geq 0\}$ be a jump process with i.i.d. interjump times $\{\delta_n = T_{n+1} - T_n,\ n \geq 0\}$ (T_n being instant of the nth jump, with $T_0 = 0$), of which size jump at time $t = T_n$ is

$$N_t(a) - N_{t-}(a) = \sum_{k=\kappa(n-1)+1}^{\kappa(n)} \sigma_k, \tag{3.1}$$

with the usual convention $\sum_{k=j}^{i} = 0$ whenever $i < j$, and where

$$\kappa(i) := \sum_{j=1}^{i} a_j$$

and $\{\sigma_n,\ n \geq 1\}$ is a stationary sequence of non negative random variables (not necessarily independent). We suppose that $a = \{a_n,\ n \geq 1\}$, $\{\delta_n = T_{n+1} - T_n,\ n \geq 0\}$ and $\{\sigma_n,\ n \geq 1\}$ are independent. Note that jumps of process $\{N_t(a),\ t \geq 0\}$ are indentically distributed but not independent, and potentially of size 0. We then define process $\{Q_t(a),\ t \geq 0\}$ that satisfies the linear equation

$$\left\{ \begin{array}{rcl} dQ_t(a) & = & dN_t(a) - \mu Q_t(a)dt \\ Q_0(a) & = & 0 \end{array} \right. \tag{3.2}$$

for some $\mu > 0$. The main problem addressed is the following : if $\{\sigma_n,\ n \geq 1\}$ and $\{\delta_n = T_{n+1} - T_n,\ n \geq 0\}$ are fixed throughout, what is the optimal stationary sequence $a = \{a_n,\ n \geq 1\}$

that solves the following problem

$$\begin{cases} \text{Minimize } \mathbb{E}(h(Q_\infty(a))) \\ \text{s.t. } \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n \geq p \end{cases} \tag{3.3}$$

for some fixed $p \in (0,1)$ and non decreasing convex function $h(.)$, where $Q_\infty(a)$ is the limiting random variable in distribution of $Q_{T_n}(a)$ as $n \to +\infty$? (the existence of this convergence in distribution will be justified in upcoming Proposition 8).

Practically, $Q_t(a)$ may be interpreted as a fluid queue with linear service rate μ. Packets of (fluid) data are of size distributed as σ_1, and arrive according to process $\{N_t(a),\ t \geq 0\}$ that satisfies (3.1). In other words, a packet of size distributed as σ_1 arriving at time T_n is either accepted if $a_n = 1$, or rejected if $a_n = 0$. Problem (3.3) aims at finding the optimal acceptance/rejection sequence minimizing the cost function $\mathbb{E}(h(Q_\infty(a)))$, such that in the long run a minimum proportion p of packet is accepted. This problem is an *open loop* problem, meaning that policy $a = \{a_n,\ n \geq 1\}$ is chosen in advance, once and for all, independently of the evolution of the queue. A result by Altman, Gaujal and Hadjek [AGH03] basically states that, provided that certain functions are *multimodular* (of which definition is given hereafter), then the sequence $a = \{a_n,\ n \geq 1\}$ that solves Problem (3.3) will turn out to be a so-called *bracket sequence*.

Before tackling the problem, we introduce definition and tools that will enable us to deal with it. We introduce the notion of multimodularity of a function (see Definition 1 p.13 of [AGH03]) :

Definition 1. *Let $e_1,\ e_n,\ s_i,\ i = 1, ..., n-1$ be vectors in $\mathbb{R}^n$ defined by $e_1 := (1, 0, ..., 0)$, $e_n := (0, ..., 0, 1)$ and $s_i := (0, ..., 0, 1, -1, 0, ..., 0)$ (with 1 on $(i-1)$-th position and -1 on i-th position), and $\mathcal{F}_n := \{e_1, -s_1, ..., -s_{n-1}, -e_n\}$. A function $f : \mathbb{N}^n \longrightarrow \mathbb{R}$ is multimodular on $\mathbb{N}^n$ if for all $x \in \mathbb{N}^n$ and all v and w in $\mathcal{F}_n$, $v \neq w$, one has*

$$f(x+v) + f(x+w) \geq f(x) + f(x+v+w).$$

Before stating the main result, we justify that for all stationary sequence $a = \{a_n,\ n \geq 1\}$, $Q_{T_n}(a)$ converges in distribution to some r.v. $W(a)$ as $n \to +\infty$. The following result is rather standard, as seen e.g. in Expression (3.13) in [AK96] or in Expression 1.8 in Chapter 2. For this result as well as for the rest of this section, it will be more convenient to consider double sided versions $a = \{a_n,\ n \in \mathbb{Z}\}$, $\{\sigma_n,\ n \in \mathbb{Z}\}$ and $\{\delta_n,\ n \in \mathbb{Z}\}$.

Proposition 8. *$Q_{T_n}(a)$ converges in distribution as $n \to \infty$ to*

$$W(a) = \int_{-\infty}^{0} \exp(\mu s) dN_s(a).$$

Note that we did not impose a specific distribution to the interarrival times δ_n, so that convergence in distribution of $Q_{T_n}(a)$ towards $W(a)$ as $n \to \infty$ does not imply convergence of $Q_t(a)$ as $t \to \infty$ towards the same limit, except, from the PASTA property, when the δ_n's are exponentially distributed i.e. when arrivals occur according to a Poisson process.

The main result is the following :

Theorem 15. *Let Θ be $\mathcal{U}([0,1])$ distributed and independent from $\{\sigma_n,\ n \geq 1\}$ and $\{\delta_n,\ n \geq 1\}$. The bracket sequence*

$$a_i = u_i(\Theta, p) := \lfloor p(i+1) + \Theta \rfloor - \lfloor pi + \Theta \rfloor, \quad i \in \mathbb{N}, \tag{3.4}$$

solves Problem (3.3).

The existence and form of the optimal sequence (3.4) comes from application of Theorem 6 p.25 of [AGH03]. In order to apply it, one however needs to verify the following points

1. that $\underline{a}_n = (a_1, ..., a_n) \mapsto \mathbb{E}(h(Q_{T_n}(\underline{a}_n)))$ is non decreasing in each a_i.
2. if $n < m$, $\mathbb{E}(h(Q_{T_n}(a_{m-n+1}, ..., a_m))) \leq \mathbb{E}(h(Q_{T_m}(a_1, ..., a_m)))$,
3. if $n < m$, $\mathbb{E}(h(Q_{T_n}(a_1, ..., a_n))) = \mathbb{E}(h(Q_{T_m}(0, ..., 0, a_1, ..., a_n)))$,
4. $\underline{a}_n = (a_1, ..., a_n) \mapsto \mathbb{E}(h(Q_{T_n}(\underline{a}_n)))$ is multimodular.

Here we use notation $Q_{T_n}(\underline{a}_n) = Q_{T_n}(a_1, ..., a_n)$ to underline that $Q_t(a)$ only depends on $(a_1, ..., a_n)$ on $t \in [0, T_n]$. Proving these properties, especially the multimodularity in Point 4., is done thanks to linearity of Equation (3.2) satisfied by $\{Q_t(a),\ t \geq 0\}$, which implies that, if $v \in \mathcal{F}_n$, $\{Q_t(a+v),\ t \geq 0\}$ is the sum of two processes that satisfy linear Equations of the form (3.2).

Another aspect in [GR07] is another optimization problem that involves how to optimally send packets of fluid data to two queues that satisfy Equation (3.2). To conclude this section, we will present how Theorem 15 translates to a risk theory framework. Let $a = \{a_n,\ n \in \mathbb{Z}\}$ and $\{N_t(a),\ t \geq 0\}$ be defined as before, and consider the following risk process

$$\left\{ \begin{array}{rcl} R_0(a) & = & u > 0, \\ dR_t(a) & = & (c + \mu R_t(a))dt - dN_t(a) \end{array} \right. \tag{3.5}$$

where $c > 0$ is the global premium rate and $\mu > 0$ is the interest force. Evolution of $R_t(a)$ is explained as for $Q_t(a)$: whenever a claim of size distributed as σ_1 occurs at time T_n, it is either taken in charge if $a_n = 1$ or rejected (for example sent to another branch of the same insurance company) if $a_n = 0$. It turns out that the ruin time defined as the first passage of $R_t(a)$ under zero is difficult to study here, so we will rather consider the definitive ruin time defined, as in Section 1.4.1, by

$$\tau(a) := \inf\{t \geq 0 |\ R_t(a) < -c/\mu\}.$$

The corresponding optimization problem is the following : If $g : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous, concave and increasing function,

$$\left\{ \begin{array}{l} \text{minimize } \displaystyle\int_0^{\infty} \mathbb{P}_{g(u)}(\tau(a) < +\infty) du \\ \text{subject to } \displaystyle\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n \geq p, \end{array} \right. \tag{3.6}$$

where $\mathbb{P}_{g(u)}(\tau(a) < +\infty)$ is the (absolute) ruin probability starting from $g(u)$. This problem is the straight counterpart of (3.3) by simply remarking that, thanks to a duality relation in the same vein as that in Proposition 5 $\int_0^{\infty} \mathbb{P}_{g(u)}(\tau(a) < +\infty) du = \mathbb{E}(h(Q_\infty(a)))$ with $h = g^{-1}$. In other words, Problem (3.6) is about minimizing a weighted functional of the ruin probability, with the constraint that a minimum proportion p of claims is accepted (that is, not refused or redispatched to some other branch) in the long run. There is freedom in choosing g, so long as it meets the requirement and, importantly, that the integral in the cost function converges. For example, one might take $g(u) = \ln(1 + u)/\beta$ for some $\beta > 0$, in order for the cost function be exponentially weighted : $\int_0^{\infty} \mathbb{P}_{g(u)}(\tau(a) < +\infty) du = \int_0^{\infty} \mathbb{P}_v(\tau(a) < +\infty) \exp(\beta v) dv$. This choice gives more importance to the ruin probability when initial reserve is large. The analog of Theorem 15 is the following :

Theorem 16. *Let Θ a random variable with uniform distribution on $[0,1]$, independent from the $\{T_{k+1}-T_k\}$ and the $\{\sigma_k\}$. $\{a_n,\ n\in\mathbb{N}\}$ defined by*

$$a_i = u_{-i}(\Theta, p) = \lfloor p(-i+1)+\Theta \rfloor - \lfloor -pi+\Theta \rfloor$$

solves (3.6).

To finish, we mention that, as in the fluid queue context, a problem is addressed in [GR07] on how to optimally dispatch incoming claims to two branches of an insurance company.

3.2 What is the amount of claim that caused ruin ?

This part concerns [RCLT13]. The motivation is the following : Let us consider the following risk process

$$\mathcal{R}_t = u + c\,t - S_t + \sigma B_t, \quad t \geq 0, \tag{3.7}$$

where u is the initial capital of the risk process, c is the premium rate received per unit time, the aggregate claim amount $\{S_t = \sum_{k=1}^{N_t} V_k,\ t\geq 0\}$ is a compound Poisson process with intensity λ, the number of claims up to time t, $\{N_t,\ t\geq 0\}$, is a Poisson process with parameter λ ($S_t = 0$ when $N_t = 0$), $V_1, V_2, \ldots$ are the independent and identically distributed jumps (claim amounts), and $\{B_t,\ t\geq 0\}$ is the standard brownian motion. [RCLT13] attempts to address the following issues :

- obtaining the joint distribution of the aggregate claim amount up to ruin time jointly to the ruin time,
- knowing whether ruin occurred thanks to "oscillation" (i.e. because of the brownian part) or by jumps.

Those will be dealt with using the following standard scheme : embedding (in the case where claims are say exponentially distributed), then devising a Lundberg equation.

The embedding process is very similar to that of Section 1.1.2. Let us suppose that the V_k's are $\mathcal{E}(\mu)$ distributed. Replacing vertical jumps by oblique lines of slope $-1/a$ yields continuous process $\{R_t = R_t^a,\ t\geq 0\}$ as illustrated on Figure 3.1. The associated Markov chain $\{J(t),\ t\geq 0\}$ has state space $\{1,2\}$, 1 being the state corresponding to evolution of the brownian motion, and state 2 to occurrence of a claim. However, in what follows some results may be, as usual, generalized to Phase type claims. We let $Q = (q_{ij})_{i,j=1,2} = \begin{pmatrix} -\lambda & \lambda \\ \mu/a & -\mu/a \end{pmatrix}$ the transition matrix of $\{J(t),\ t\geq 0\}$. Ruin times for both processes are denoted by

$$\mathcal{T} := \inf\{t\geq 0 \,|\, \mathcal{R}_t \leq 0\}, \quad \tau = \tau_a := \inf\{t \geq 0 \,|\, R_t \leq 0\}.$$

Similarly to Proposition 4, relation between $(\mathcal{T}, S_{\mathcal{T}})$ and $(\tau, J(\tau))$ is given by

Proposition 9. *For all $q\geq 0$ and $a>0$, we have*

$$\mathbb{E}[\, e^{-q(\mathcal{T}+a S_{\mathcal{T}})} \,|\, \mathcal{R}_{\mathcal{T}} = 0,\ \mathcal{T} < +\infty\,] = \mathbb{E}[\, e^{-q\tau} \,|\, J(\tau) = 1,\ \tau < +\infty\,]$$

and

$$\mathbb{E}[\, e^{-q(\mathcal{T}+a S_{\mathcal{T}})} \,|\, \mathcal{R}_{\mathcal{T}} < 0,\ \mathcal{T} < +\infty\,] = \frac{\mu}{\mu+aq}\mathbb{E}[\, e^{-q\tau} \,|\, J(\tau) = 2,\ \tau < +\infty\,].$$

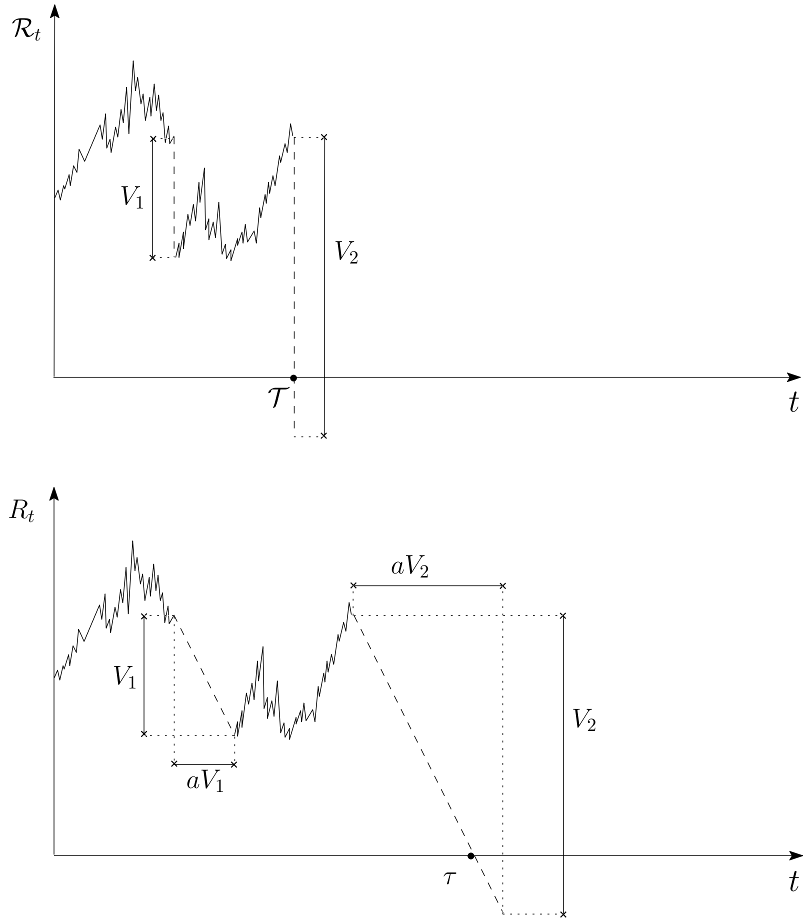

FIGURE 3.1 – Embedding

The above Proposition imply that Laplace transform $\mathbb{E}[\, e^{-\alpha\,\mathcal{T}-\beta\, S_{\mathcal{T}}} \,|\, A,\, \mathcal{T} < +\infty\,]$, with $A = [\mathcal{R}_{\mathcal{T}} = 0]$ (ruin by oscillation) or $[\mathcal{R}_{\mathcal{T}} < 0]$ (ruin through a jump) for all positive α and β is available simply by setting $q = \alpha$ and $a = \beta/\alpha$. Thus we are interested from now on to joint distribution of $(\tau, J(\tau))$, i.e. for example in quantities of the form

$$\mathbb{E}[e^{-q\tau}\varphi(J(\tau))] \tag{3.8}$$

for a large class of $\varphi(.)$. Note that this is where things are a bit different from setting in Section 1.1.2, where only τ is considered. First remark that generator of Markov process $\{(R_t, J(t)),\ t \geq 0\}$ is

$$\mathcal{A}f(x,i) = \frac{\sigma(i)^2}{2} f''(x,i) + h(i)f'(x,i) + \sum_{j=1,2} q_{i,j} f(x,j)$$

where $f(.,i)$ is twice differentiable, $i = 1,2$. $\sigma(.)$ and $h(.)$ come from the embedding process and and are such that $\sigma(1) = \sigma$, $\sigma(2) = 0$, $h(1) = c$, $h(2) = -1/a$. This may be written in matrix form

$$\mathcal{A}f(x) = Sf''(x) + Hf'(x) + Qf(x)$$

where $S := \text{diag}\left[\sigma(1)^2/2, \sigma(2)^2/2\right] = \text{diag}\left[\sigma^2/2, 0\right]$, $H = H_a := \text{diag}[h(1), h(2)] = \text{diag}[c, -1/a]$, and $f(x) := (f(x,1), f(x,2))'$. The following result shows that finding the eigenvector of $\mathcal{A}$ associated to eigenvalue q can contribute to computing quantity (3.8).

Lemma 3. *For all $q \geq 0$, let f be a solution to the following matrix equation*

$$\mathcal{A}f(x) = qf(x). \tag{3.9}$$

such that $\lim_{x\to+\infty} f(x) = 0$ (and, implicitly, that $f(x,i)$ is twice differentiable with respect to x). Then we have $\mathbb{E}\left[e^{-q\tau} f(0, J(\tau))1_{\{\tau<+\infty\}}\right] = f(u, J(0)) = f(u, 1)$.

This lemma bears some similarity with Theorem 2.1 (ii) of [PG97], with the difference that no boundary condition are required at $x = 0$ (in fact, the boundary condition appears in $f(0, J(\tau))$). Before proceeding further with Equation (3.9), let us see how Lemma 3 can be applied in order to obtain (3.8). Let us suppose that f verifies in addition $f(0,j) = 1_{\{j=1\}}$, $j = 1, 2$, (i.e. $f(0) = (1,0)'$), then Lemma 3 reads

$$\mathbb{E}[e^{-q\tau}1_{\{J(\tau)=1,\,\tau<+\infty\}}] = f(u, J(0)) = f(u, 1).$$

Similarly, taking solution f to (3.9) such that $f(0) = (0,1)'$ yields $\mathbb{E}[e^{-q\tau}1_{\{J(\tau)=2,\,\tau<+\infty\}}] = f(u, J(0))$, and setting the initial condition $f(0) = (1,1)'$ gives $\mathbb{E}[e^{-q\tau}1_{\{\tau<+\infty\}}] = f(u, J(0))$, the Laplace transform of the time of ruin.

Let us now focus on Equation (3.9) with condition $\lim_{x\to+\infty} f(x) = 0$. This equation can be written in matrix form as

$$Sf''(x) + Hf'(x) + (Q - qI)f(x) = 0. \tag{3.10}$$

which is closely linked to the *Lundberg equation*

$$\det(z^2S + zH + Q - qI) = \begin{vmatrix} \sigma^2z^2/2 + cz - \lambda - q & \lambda \\ \mu/a & -z/a - \mu/a - q \end{vmatrix} = 0 \tag{3.11}$$

which shows up in many papers (see [BB08, RL09]). In fact, solutions to (3.10) are of the form

$$f(x) = \sum_{i=1}^{3} a_i e^{z_i x}\phi_i,$$

where the a_i's are scalars, z_1, z_2, z_3 are the three solutions to (3.11) (that we suppose are distinct, although there is a discussion in [RCLT13] on how all subsequent results are modified in case of multiple roots), and ϕ_1 and ϕ_2 are the corresponding eigenvectors satisfying

$$(z_i^2S + z_iH + Q - qI)\,\phi_i = 0.$$

To identify which of these solutions fit requirement $\lim_{x\to+\infty} f(x) = 0$ in Lemma 3, one needs to censor out whichever solution z_i, $i = 1, 2, 3$, has positive real part, i.e. needs to know location of roots of Lundberg equation (3.11). The following proposition addresses this issue and says how to obtain solutions f satisfying (3.11) with the vanishing at infinity condition $\lim_{x\to+\infty} f(x) = 0$.

Proposition 10. *Equation (3.11) has exactly two roots z_1 and z_2 with negative real part and one real non-negative root z_3. Besides, in the case where $a > 0$ and $q \geq 0$ are such that z_1 and z_2 are* *distinct (multiplicity equal to* 1*),* (3.9) *admits a unique solution* f *satisfying* $\lim_{x\to+\infty} f(x) = 0$ *with any fixed boundary condition* $f(0) = (f(0,1), f(0,2))'$*.*

Some remarks concerning the above result :

- In the case where claims and interclaim times are Phase type distributed with respective phases n_+ and n_- then (3.10) involves $\mathbb{R}^{n\times n}$ matrices with $n = n_+ + n_-$. In that case Proposition 10 generalizes to a result where one needs to locate solutions of the corresponding Lundberg equation (3.11), which this times admits $2n_+ + n_-$ solutions. This problem is already present in various problems in queueing or risk theory where there is Markov modulation, and is solved thanks to a modification of Gershgorin's disc theorem, see e.g. [KK95] for the case $q = 0$.
- A similar approach mentioned in [RCLT13] is used for considering the double sided exit problem, which in risk theory is translated as a ruin problem of a risk process with dividend.
- There seems to have been substantial improvement concerning determination of hitting time distribution of Markov additive process which may complete or even generalize some results in [RCLT13]. In particular, one major result in [IP12] is determination of Laplace transforms of hitting times of a Markov additive process in terms of its corresponding scale function.

To conclude this section, we discuss what happens when claims are no longer exponentially or Phase type distributed but have a density $p(.)$. Embedding is now irrelevant. However one can hope to use standard tools in risk theory and expect to obtain an integrodifferential equation for the Laplace transform of ruin time jointly to the associated aggregated claim. We are only interested in ruin by oscillation

$$\phi_d(u) := \mathbb{E}[\, e^{-\alpha\mathcal{T}-\beta S_{\mathcal{T}}} 1_{\{\mathcal{T}<+\infty,\mathcal{R}_{\mathcal{T}}=0\}}].$$

Theorem 17. *ϕ_d is a twice differentiable function on $(0,+\infty)$. It verifies the integro-differential equation*

$$\frac{\sigma^2}{2}\,\phi_d''(u) + c\,\phi_d'(u) + \lambda\int_0^u e^{-\beta x}\phi_d(u-x)\,p(x)\,dx = (\lambda+\alpha)\,\phi_d(u), \quad u>0, \tag{3.12}$$

and the renewal equation

$$\phi_d(u) = \int_0^u \phi_d(u-y)g(y)dy + e^{-b\,u}, \tag{3.13}$$

where $\rho = \rho(\alpha,\beta)$ is the unique positive root of generalized Lundberg equation

$$\lambda\int_0^\infty e^{-(\rho+\beta)u}p(u)du = \lambda+\alpha-c\rho-\frac{\sigma^2}{2}\rho^2, \tag{3.14}$$

$b := 2c/\sigma^2+\rho$, and g is a function that depends on paramters of the model and density of claims.

Although (3.12), (3.13), (3.14) are very close to standard equations, especially in the presence of a perturbating brownian motion (see [Tsa01, TW02]), there are all the same some changes due to presence of the aggregate claim at time of ruin (which translates as presence of factor β in the Lundberg equation (3.14)). The reason why integro differential equation concerns ruin by oscillation and not by jump is that, in the latter case, we could not prove that the corresponding Laplace transform in the latter case is indeed twice differentiable. Note that proving sufficient regularity for the ruin probability or Laplace transforms in risk models is not always straightforward, and is one of the assumptions sometimes made prior to deriving integro differential equations (see Theorem

2.1 (i) of [PG97]). The approach adopted here is to first establish a renewal equation for ϕ_d then to prove that the integrand has sufficient regular properties; this approach bears some similarities with earlier papers, see e.g. Equation (2.3) of Theorem 2.1 as well as Theorems 2.2 and 2.3 in [WW01].

3.3 Ruin time of a Wong Pearson diffusion risk process

We consider in [ALR09] the following model

$$dX_t = c(X_t)dt + \sigma(X_t)dB_t - dS_t. \tag{3.15}$$

$S_t := \sum_{i=1}^{N_t} Z_i$ is the aggregate claim amount. $\{N_t,\ t \geq 0\}$ is a Poisson process with intensity λ, $\{B_t,\ t \geq 0\}$ is a brownian motion, and the Z_i's are i.i.d. claim sizes with cdf $F(.)$, density $f(.)$ and first moment m_1 (which we suppose exists). Drift and diffusion coefficients are given by

$$\begin{aligned} c(x) &= p + rx \\ \sigma(x) &= \sqrt{\sigma_0^2 + \sigma_1^2 x + \sigma_2^2 x^2}. \end{aligned}$$

p and r respectively represent the premium rate and interest force. X_t is defined on a random interval such that $\sigma_0^2 + \sigma_1^2 X_t + \sigma_2^2 X_t^2 \geq 0$ on that interval. This kind of diffusion with (negative) jumps is the so called *generalized Wong Pearson diffusion*, characterized by the fact that $c(.)$ is an affine function and $\sigma(.)^2$ is quadratic. This model covers a wide range (some of which will be focused on later on) such as

- the generalized Black Scholes model with paramaters $c(x) = rx$ and $\sigma(x) = \sigma_2 x$,
- the generalized Ornstein-Uhlenbeck (GOU) process with parameters $c(x) = p + rx$ and $\sigma(x) = \sigma_0 > 0$.
- the generalized Cox Ingersoll Ross (GCIR) model with $c(x) = p + rx$ and $\sigma(x) = \sigma_1\sqrt{x}$.

In the following we will use the more convenient notation

$$d(x) = \frac{\sigma^2(x)}{2} = \frac{\sigma_0^2 + \sigma_1^2 x + \sigma_2^2 x^2}{2} = d_0 + d_1 x + d_2 x^2,$$

and we define the following quantities :

- the ruin time of $\{X_t,\ t \geq 0\}$:

$$\tau := \inf\{t \geq 0 : X_t < 0\},$$

- the probability of eventual ruin starting from $X_0 = x$ and the survival probability

$$\psi(x) := \mathbb{P}_x(\tau < +\infty), \quad \bar{\psi}(x) := 1 - \psi(x),$$

- the Laplace transform of τ starting from x

$$\psi_q(x) := \mathbb{E}_x(e^{-q\tau} 1_{\{\tau < +\infty\}}).$$

The aim of [ALR09] is to identify in which case it is possible to expand $\psi(x)$ or $\psi_q(x)$ in the following form

$$\bar{\psi}(x) = \sum_{n=0}^{\infty} a_n E(n, \alpha x), \quad \bar{\psi}_q(x) = 1 - \psi_q(x) = \sum_{n=0}^{\infty} a_n^*(q) E(n, \alpha x), \tag{3.16}$$

for some real valued sequences $(a_n)_{n\in\mathbb{N}}$ and $(a_n(q))_{n\in\mathbb{N}}$, and where $\bar{E}(n, \alpha x)$ is the *Erlang complementary cumulant distribution function* defined as

$$\begin{aligned}\bar{E}(n, \alpha x) &= \mathbb{P}\left\{\sum_{i=1}^{n} \mathcal{E}_i(\alpha) > x\right\} = \frac{1}{\Gamma(n)} \int_{\alpha x}^{\infty} y^{n-1} e^{-y} dy \\ &= e^{-\alpha x} \sum_{j=0}^{n-1} \frac{(\alpha x)^j}{j!} = 1 - E(n, \alpha x), \ n \in \mathbb{N},\end{aligned}$$

where $\sum_{i=1}^{n} \mathcal{E}_i(\alpha), n \geq 1$ is a random variable with an Erlang distribution, that is a sum of independent exponential random variables $\mathcal{E}_i(\alpha)$, $i = 1, \ldots, n$ with parameter $\alpha > 0$ (and $\bar{E}(0, \alpha x) = 0$), and $\Gamma(x)$ is the Gamma integral function $\displaystyle\int_0^{\infty} t^{x-1} e^{-t} dt$.

Expansions of the form (3.16) are investigated for the cdf of ruin time τ starting from x in [Tay78] for the case $c(x) = p$ constant and $\sigma(x) = 0$, and in [ATT01] for $c(x) = p + rx$ and $\sigma(x) = 0$. It is motivated by the fact that the set of functions $\{x \mapsto E(n, \alpha x), \ n \in \mathbb{N}\}$ is closed by convolution, i.e.

$$\int_0^x E(n, \alpha(x-y)) E(m, \alpha\, dy) = E(n+m, \alpha x), \quad n, m \in \mathbb{N}. \tag{3.17}$$

Besides, $x^k \frac{\partial}{\partial x} E(n, \alpha x)$, $k = 0, 1$, $x^k \frac{\partial^2}{\partial x^2} E(n, \alpha x)$, $k = 0, 1, 2$, can all be expressed in terms of $E(j, \alpha x)$ for $j = 0, 1, 2$. These properties are particularly important, since ψ and ψ_q verify the following integro differential equations

$$0 = c(x)\psi'(x) + d(x)\psi''(x) + \int_0^{\infty} [\psi(x-z) - \psi(x)] \lambda f(z) dz := \mathcal{G}\psi(x), \tag{3.18}$$

$$q\psi_q(x) = c(x)\psi_q'(x) + d(x)\psi_q''(x) + \int_0^{\infty} [\psi_q(x-z) - \psi_q(x)] \lambda f(z) dz = \mathcal{G}\psi_q(x) \tag{3.19}$$

These equations are very standard in risk theory, and require some extra boundary and regularity conditions not mentioned here, which will be made clear in the particular cases studied in the following (Note that (3.19) is the same as (3.9)). In order for (3.16) to be obtained, one needs however to focus on several technical points. For one thing, one needs to make clear what boundary conditions accompany (3.18) and (3.19) in order to characterize the finite ruin probability and Laplace transform. Then one needs to check that an expansion (3.16) satisfies these conditions, and is a convergent series. This indeed will not be always the case, and will depend on the different parameters. The next subsections present some cases where everything turns out fine, in the case of exponentially distributed jumps.

Prior to that, we present two general results. One concerns the so-called transience condition of the risk process (3.15), which states that $\lim_{x\to\infty}\bar{\psi}(x) = 1$, an essential boundary condition. In the case of such a model with no jump, these conditions are well known and are the following :

(C1) $d(x) = d_0$ and $p > 0$ (Brownian motion).
(C2) $d(x) = d_2x^2$ and $r > d_2$ (Black-Scholes).
(C3) $d(x) = d_0$ and $r > 0$ (Ornstein Uhlenbeck).
(C4) $d(x) = d_1x$ and $r > 0$ (CIR).

In the case of jumps, it is not obvious that such conditions are sufficient, in particular in view of the brownian case, where condition turns out to be $p > \lambda m_1$ (positive safety loading). [Pau98] proves that Condition (C2) is sufficient in the generalized Black-Scholes case, see also [FKP02]. The following result states that this is true in the case of exponentially distributed jumps.

Proposition 11. *Let the risk process X_t satisfy (3.15) with the Z_n (jumps) being $\mathcal{E}(\alpha)$ distributed. We have $\bar{\psi}(x) \longrightarrow 1$ as $x \to +\infty$ in the following cases :*

(C2) *$r > d_2$ in the Generalised Black-Scholes model $d(x) = d_2x^2$.*
(C3) *$r > 0$ in the Generalised Ornstein Uhlenbeck model $d(x) = d_0$.*
(C4) *$r > 0$ in the Generalised CIR model $d(x) = d_1x$.*

The second result is much more computational, and presents necessary relations satisfied by sequences $(a_n)_{n\in\mathbb{N}}$ and $(a_n(q))_{n\in\mathbb{N}}$ if ψ and ψ_q have expansions (3.16) and satisfy (3.18) and (3.19), using (3.17) and relations with derivatives of the Erlang distribution function, in a general case when the cdf of the Z_n's is expressed in terms of the $E(n, \alpha x)$.

Theorem 18. *Assume that the claim distribution function admits an Erlang expansion*

$$F(y) = \mathbb{P}\{Z_i \le y\} = \sum_{n=1}^{N} f_n E(n, \alpha y), \quad \text{where} \ \sum_{n=1}^{N} f_n = 1,$$

where N may be infinite. Then :

1. *coefficients $(a_n)_{n\in\mathbb{N}}$ satisfy the following relations :*

$$\underline{n=0}: \quad d_0\alpha^2 a_2 + a_1(p\alpha - d_0\alpha^2) - \lambda a_0 = 0 \tag{3.20}$$

$$\begin{aligned} \underline{n\ge 1}: \quad & d_0\alpha^2 a_{n+2} + a_{n+1}(p\alpha - 2d_0\alpha^2 + d_1\alpha n) \\ & + a_n(rn - p\alpha + d_0\alpha^2 - (2n-1)d_1\alpha + d_2 n(n-1) - \lambda) \\ & + a_{n-1}(-r(n-1) + d_1\alpha(n-1) - 2d_2(n-1)^2) \\ & + d_2(n-1)(n-2)a_{n-2} = -\lambda \sum_{i=1}^{\min(n,N)} f_i\, a_{n-i} \end{aligned} \tag{3.21}$$

2. *coefficients $(a_n^* = a_n^*(q))_{n\in\mathbb{N}}$ satisfy the same recurrence (3.20), (3.21), but adding qa_n^* −*

$q1_{\{n=0\}}$ *to the right-hand side :*

$$\begin{aligned}
\underline{n=0}: \quad & d_0\alpha^2 a_2^* + a_1^*(p\alpha - d_0\alpha^2) - \lambda a_0^* = q a_0^* - q && (3.22)\\
\underline{n \geq 1}: \quad & d_0\alpha^2 a_{n+2}^* + a_{n+1}^*(p\alpha - 2d_0\alpha^2 + d_1\alpha n) + \\
& a_n^*(rn - p\alpha + d_0\alpha^2 - (2n-1)d_1\alpha + d_2 n(n-1) - \lambda) \\
& + a_{n-1}^*(-r(n-1) + d_1\alpha(n-1) - 2d_2(n-1)^2) \\
& + d_2(n-1)(n-2)a_{n-2}^* = -\lambda \sum_{i=1}^{\min(n,N)} f_i\, a_{n-i}^* + q a_n^* && (3.23)
\end{aligned}$$

As some parameters among p, r, d_0, d_1, d_2 will vanish in what follows, relations (3.20), (3.21), (3.22) and (3.23) will greatly simplify.

3.3.1 Generalized Ornstein-Uhlenbeck process, and Brownian motion with jumps.

We consider here case

$$c(x) = p + rx, \quad d(x) = d_0 > 0,$$

with $Z_1 \sim \mathcal{E}(\alpha)$ and $r \geq 0$. In that case, Theorem 2.1 (i) of [PG97] says that a bounded twice differentiable solution to the integro differential equation (3.18) with $\psi(0) = 1$ and $\lim_{x\to+\infty} \psi(x) = 0$ is the probability of eventual ruin. Therefore sufficient conditions for an expansion of $\psi(x)$ of the form (3.16) are

- that $\sum_{n=0}^{\infty} |a_n| < +\infty$, i.e. that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent,
- that the following normalization holds

$$\lim_{x\to+\infty} \bar{\psi}(x) = \sum_{k=0}^{\infty} a_k = 1. \tag{3.24}$$

  (the limit being justified because of the absolute convergence of the series), as justified by Proposition 11,
- that

$$a_0 = \bar{\psi}(0) = 0.$$

Note that boundedness and differentiability will be automatically satisfied, again by absolute convergence of $\sum_{n=0}^{\infty} a_n$. Concerning expression of coefficients, Theorem 18 simplifies to

Lemma 4. *In the case of $\mathcal{E}(\alpha)$ distributed jumps, $(a_n)_{n\in\mathbb{N}}$ is expressed recursively in the following manner :*

$$a_{n+2} = \left(1 - \frac{p}{d_0\alpha}\right) a_{n+1} + \frac{\lambda - rn}{d_0\alpha^2} a_n. \tag{3.25}$$

A sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (3.25) obviously does not always meet the absolute convergence of its series requirement. We present two cases where everything works out fine.

A finite expansion in the Generalized Ornstein-Uhlenbeck model . Let us suppose that

$$p = d_0\alpha \quad \text{and} \quad \lambda = r(2N+1) \text{ for some } N \geq 0. \tag{3.26}$$

The first condition ensures that $a_n = 0$ for n even, thanks to Lemma 4 and the fact that $a_0 = 0$. The second condition yields that $a_n = 0$ as soon as $n \geq 2N+3$, so that expansion is for odd indexes and finite. The expression of a_n for $n = 2k+1$ is thus given by

$$a_{2k+1} = a_1 \prod_{i=1}^{k} \frac{\lambda - r(2i-1)}{d_0\alpha^2}, \quad 0 < k < N.$$

Note that Condition (3.26) resembles Condition $\lambda = rN$ for some $N \in \mathbb{N}$ of Theorem 1 of [ATT01] in which authors express the cdf of the ruin time, not just probability of eventual ruin, and where a finite expansion is established. Indeed, when $\lambda = rN$ and $d_0 = 0$ then one can express a_{n+1} in function of a_n, then prove that $a_n = 0$ for $n \geq N+1$, by considering (3.26) multiplied by d_0.
Condition (3.26), which enables Erlang expansions of the ruin probability, is of course not always satisfied. However it yields upper and lower bounds for the survival probability. For example let us suppose that $p = d_0\alpha$ but that $\frac{\lambda}{r} \notin 2\mathbb{N}+1$. Let us set $N := \lfloor \frac{1}{2}\left(\frac{\lambda}{r} - 1\right) \rfloor$, $\lambda^N := r(2N+1)$ and $\lambda^{N+1} := r(2(N+1)+1)$. Then the probabilities of survival $\bar{\psi}^N$ and $\bar{\psi}^{N+1}$ associated to arrival rates of claims λ^N and λ^{N+1} have each a finite Erlang expansion and provide upper and lower bounds of $\bar{\psi}$:

$$\bar{\psi}^N(x) \geq \bar{\psi}(x) \geq \bar{\psi}^{N+1}(x).$$

Likewise, if say $p > d_0\alpha$ then one can get upper and lower Erlang expanded bounds for $\bar{\psi}(x)$ repsecively by considering the same model but with mean claim sizes $\alpha_0 := p/d_0$ and premium rate $p_0 := d_0\alpha$. Of course the resulting bounds may not be tight, but they are very easy to compute.
An infinite expansion in the brownian motion case. When $r = 0$ then the solution of (3.25) is of the form

$$a_n = C_1 r_1^n + C_2 r_2^n$$

for some constants C_1 and C_2, and where r_1 and r_2 are the two roots of equation $X^2 - \left(1 - \frac{p}{d_0\alpha}\right) X - \frac{\lambda}{d_0\alpha^2} = 0$. If λ is small enough, or if α is large enough, then one can check that r_1 and r_2 lie in $(-1,1)$, so that $\sum_{n=0}^{\infty} |a_n| < +\infty$. Constants C_1 and C_2 are determined from $a_0 = C_1 + C_2 = 0$ as well as normalization (3.24), which translates as $1 = \frac{C_1}{1-r_1} + \frac{C_2}{1-r_2}$.

3.3.2 A finite expansion for an affine process.

We consider here case

$$c(x) = p + rx, \quad d(x) = d_0 + d_1 x,$$

with $d_0, d_1 > 0$. We suppose that there is no jump, i.e. $\lambda = 0$. Since Theorem 2.1 of [PG97] does not cover this case, it is conjectured that if a function is a twice differentiable solution to the integro differential equation (3.18) with $\psi(0) = 1$ and $\lim_{x\to+\infty} \psi(x) = 0$ then it is the probability of eventual ruin (note in particular that condition $\psi(0) = 1$ is due to $d_0 > 0$).Therefore a candidate expansion of the form (3.16) for $\psi(x)$ should verify, as in the previous subsection, that $\sum_{n=0}^{\infty} |a_n| < +\infty$, $a_0 = 0$ as well as normalization condition (3.24).

Theorem 18 simplifies to

Lemma 5. *$(a_n)_{n\in\mathbb{N}}$ is expressed recursively in the following manner :*

$$a_{n+2} = \frac{-p + d_0\alpha - d_1 n}{d_0\alpha} a_{n+1} + \frac{-rn + \alpha d_1 n}{d_0\alpha^2} a_n. \tag{3.27}$$

Since there is no jump, $\alpha > 0$ is here a free parameter. We note that a finite expansion is possible if

$$-p + \frac{d_0}{d_1} r = d_1 N \text{ for some integer } N. \tag{3.28}$$

Indeed, choosing $\alpha = r/d_1$ yields $\bar{\psi}(x) = \sum_{n=1}^{N+1} a_n E(n, r/d_1.x)$ with

$$a_n = \frac{1}{(d_0 r/d_1)^{n-1}} \prod_{k=2}^{n} [-p + d_0 r/d_1 - d_1(k-2)] a_1, \quad n \geq 2,$$

with a_1 obtained from normalization (3.24). Again, Condition (3.28) is not always verified in practise, but upper and lower bounds for $\bar{\psi}(x)$ that have an Erlang expansion can be provided, by using the same trick as in the Generalized Ornstein Uhlenbeck case in subsection 3.3.1, using $N := \lfloor \frac{1}{d_1} \left(-p + \frac{d_0}{d_1} r \right) \rfloor$, and premium rates $p_N := \frac{d_0}{d_1} r - d_1 N$ and $p_{N+1} := \frac{d_0}{d_1} r - d_1(N+1)$.

3.3.3 Expansion of the Laplace transform for a generalized CIR process.

We consider

$$c(x) = p + rx, \quad d(x) = d_1 x,$$

with $Z_1 \sim \mathcal{E}(\alpha)$. We focus here on trying to find in what condition an expansion (3.16) of $\psi_q(x)$ is possible. Since [PG97] is again not applicable, we start by proving the following result

Lemma 6. *Let $q \in (0, +\infty)$. If $\psi_q(x)$ is twice differentiable and satisfies*

1. *ψ_q and ψ'_q have logarithmic growth, i.e. $|\psi_q(x)|$ and $|\psi'_q(x)|$ being $O(\ln(x))$,*
2. *$\psi_q(x) = \psi_q(0) = 1$ if $x < 0$,*
3. *ψ_q satisfies (3.19) on $x \in [0, +\infty)$,*

then $\psi_q(x) = \mathbb{E}_x(e^{-q\tau})$.

Note that Condition $\psi_q(x) = 1$ if $x < 0$ is a bit artificial and is here for technicalities. We also note that when $x = 0$ then ruin occurs at time $t = 0$; this is typical of the CIR process and is different from e.g. the geometric brownian case $d(x) = d_2 x^2$. The objective is to prove that en expansion of $\psi_q(x)$ verifies points 1. Theorem 18 simplifies and we obtain the following expansion result

Theorem 19. *$a_n^*(q) = a_n^*$ is defined by*

$$\begin{cases} a_0^* &= 0 \\ a_1^* &= \frac{-q}{p\alpha} \\ a_{n+1}^* &= \left(1 - \gamma_n + \frac{q-r}{\alpha(p+d_1 n)}\right) a_n^* + \gamma_n a_{n-1}^* \end{cases} \tag{3.29}$$

with $\gamma_n := \frac{-\lambda + (r - d_1\alpha)(n-1)}{\alpha(p+d_1 n)}$. Furthemore, under Condition

$$0 < r - d_1\alpha < d_1\alpha, \tag{3.30}$$

$\bar{\psi}_q(x) = 1 - \psi_q(x) = 1 - \mathbb{E}_x(e^{-q\tau})$ has the Erlang expansion (3.16) for $q \in (0, r]$.

The key for proving this theorem is to prove that, with $(a_n^*(q))_{n\in\mathbb{N}}$ defined as (3.29), $x \mapsto \sum_{n=0}^{\infty} a_n^*(q)E(n,\alpha x)$ has logarithmic growth, so that Lemma 6 can be applied. Let us remark that expansion (3.16) is valid on an interval $q \in (0,r]$ that does not contain 0. Indeed, proof of Theorem 19 does not seem to work for $q=0$.

3.4 Risk theory and reliability

We present here [PR13]. We consider a Lévy process $\{D_t,\ t\geq 0\}$ of the form

$$\forall t \geq 0\,,\ D_t = G_t + \sigma B_t \tag{3.31}$$

where $\{G_t,\ t\geq 0\}$ is a subordinator, i.e. a Lévy process with non decreasing sample paths, and $\{B_t,\ t\geq 0\}$ is an independent brownian motion. The objective of [PR13] is to determine the following quantities

$$\phi_w(\delta,b) = \mathbb{E}(e^{-\delta T_b} w(D_{T_b-}, D_{T_b})), \tag{3.32}$$

$$\mathbb{P}(L_b < t), \tag{3.33}$$

$$\mathbb{E}[e^{-\delta L_b} 1_{\{b-D_{L_b-}\in dy,\ D_{L_b}-b\in dw\}}], \tag{3.34}$$

$$\mathbb{E}\left[e^{-\delta L_b^*}\right], \tag{3.35}$$

for all $\delta\geq 0$, $b\geq 0$, $y\geq 0$, $w\geq 0$, where $w:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a so called *penalty function*, and

$$T_b := \inf\{t\geq 0\,;\ D_t \geq b\}$$

is the first passage time above level b of $\{D_t,\ t\geq 0\}$,

$$L_b := \sup\{0\leq u|\ D_u\leq b\} \quad \text{and} \quad L_b^* := \sup\{0\leq u|\ D_u^*\leq b\}$$

are last passage times below b of processes $\{D_t,\ t\geq 0\}$ and its reflected $\{D_t^*,\ t\geq 0\}$ defined by

$$\forall t\geq 0, \quad D_t^* := D_t - \inf_{0\leq s\leq t}(D_s\wedge 0). \tag{3.36}$$

These kind of problems are obviously related to risk theory, and are motivated this time by reliability. More precisely, D_t represents degradation state of a certain component or system at instant t. Traditionally, one consider that this component is deteriorated when it reaches a certain level $b>0$ for the first time (see e.g. [PP05] in the simple case when G_t is a gamma process and $\sigma=0$, with a more statistical oriented study). However, a recent suggestion by Barker and Newby [BN09] is to rather consider the last passage time of the process as the degradation time. This has a nice interpretation in reliability, as even if $\{D_t,\ t\geq 0\}$ reaches and goes beyond b, resulting in a temporarily degraded state of the device, it can still always recover by getting back below b provided this was not the last passage time through b. On the other hand, if this is the last passage time then no recovery is possible afterwards.

We first introduce some notation. The Laplace exponent of $\{D_t,\ t\geq 0\}$ is denoted by φ_D and verifies

$$\forall u\in\mathbb{R}, \quad e^{t\varphi_D(u)} = \mathbb{E}[e^{-uD_t}] = \exp(t\varphi_G(u))\exp(t\varphi_B(u)),$$

where φ_G and φ_B are the Laplace exponents of $\{G_t,\ t \geq 0\}$ and $\{B_t,\ t \geq 0\}$ and are such that (remember that G_t is a subordinator)

$$\varphi_B(u) = \frac{1}{2}u^2\sigma^2, \quad \varphi_G(u) = -\mu u + \int_0^\infty [e^{-ux} - 1]Q(dx),$$

for some $\mu \geq 0$, where $Q(.)$ is a measure with support in $(0, +\infty)$ (see Section 2.6.2 of [Kyp06]).

3.4.1 First passage time.

We first focus on T_b. We first note that what we are interesting in exact formulas for (3.32). However, interesting properties were discovered about the behavior of joint distribution of (T_b, D_{T_b-}, D_{T_b}) as $b \to +\infty$ in [RVV08], where the authors prove that, surprisingly, T_b becomes asymptotically independent from (D_{T_b-}, D_{T_b}), and give an explicit expresion of the Laplace transform of the correctly renormalized limiting distribution of the triplet.

We aim at confronting two approaches for determining (3.32). One is the use of *scale functions*, which were already introduced previously in Section 2.2 (see Formula (2.11)).

Definition 2. *We define for all $\delta \geq 0$ the scale function $W^{(\delta)}$ of process $\{D_t,\ t \geq 0\}$ through the following Laplace transform*

$$\int_0^\infty e^{-\lambda x} W^{(\delta)}(x)dx \;=\; \frac{1}{\varphi_D(\lambda) - \delta}, \quad \lambda > \rho(\delta), \tag{3.37}$$

where $\rho(\delta)$ is solution of the Lundberg equation

$$\delta - \frac{\sigma^2}{2}\rho^2 = \varphi_G(\rho) \quad \Longleftrightarrow \quad \delta = \varphi_D(\rho). \tag{3.38}$$

It turns out that the following result from [BK10] gives a bit more than expression (3.32).

Theorem 20 (Theorem 1 [BK10])**.** *Let us define the last maximum before hitting time T_b as $\overline{D}_{T_b-} := \sup_{t<T_b} D_t$. Then*

$$\mathbb{E}\left[e^{-\delta T_b} w(D_{T_b-}, D_{T_b}, \overline{D}_{T_b-})\right] = \int_{(0,+\infty)^3} 1_{\{v \geq y\}} w(u+b, -v-b, -y-b) K_b^{(\delta)}(du, dv, dy), \tag{3.39}$$

where function $w(.,.,.)$ verifies $w(., b, .) = 0$ and

$$K_b^{(\delta)}(du, dv, dy) := e^{-\rho(\delta)(v-y)} \left[W^{(\delta)\prime}(b-y) - \rho(\delta)W^{(\delta)}(b-y)\right] Q(du+v)dydv.$$

In particular :

$$\phi_w(\delta, b) = \int_{(0,+\infty)^2} w(u+b, -v-b)\tilde{K}_b^{(\delta)}(v)Q(du+v)dv \tag{3.40}$$

where $\tilde{K}_b^{(\delta)}(v) := \displaystyle\int_{y=0}^{v} e^{-\rho(\delta)(v-y)} \left[W^{(\delta)\prime}(b-y) - \rho(\delta)W^{(\delta)}(b-y)\right] dy.$

Several remarks spring to mind while reading this result. First, $K_b^{(\delta)}(du, dv, dy)$ requires implicitly that $W^{(\delta)}$ be differentiable : a sufficient condition for this is that D_t has unbounded variation, which is the case here since we suppose $\sigma > 0$. Second, the computational issue of how to obtain $W^{(\delta)}$ in practice is raised, so as to get closed forms for quantities (3.39) and (3.40). Recent papers deal with this practical aspect and give expressions of this scale function when $\{G_t,\ t \geq 0\}$ is a compound Poisson process with Phase type distributed jumps, or more generally when it is meromorphic, see [HK11, MK13, EY12], or give approximations of those scale functions. It is not known at the moment how to have an expression for $W^{(\delta)}$ when $\{G_t,\ t \geq 0\}$ is for example a Gamma process, which is a process that is traditionally used in reliability for modelling degradation.

This leads to the second approach for determining expression of (3.32). By using the fact that process $\{G_t,\ t \geq 0\}$ is the (pointwise) limit of a sequence of compound Poisson processes with jumps of which is related to measure $Q(.)$, we arrive at the following result using a method similar to [GM06] (we recall that that convolution of two functions f and g defined from $[0, +\infty)$ to $\mathbb{R}$ is defined by $f \star g(z) = \int_0^z f(x)g(z-x)dx$).

Proposition 12. *Let* $\omega(x) := \int_x^\infty w(x, y-x)Q(dy)$. *Function* $\phi(\delta, \cdot) = \phi_w(\delta, \cdot)$ *satisfies the renewal equation*

$$\phi(\delta, b) = \phi(\delta, \cdot) \star g(\delta, \cdot)(b) + h(\delta, b) \tag{3.41}$$

where functions $g(\cdot,\cdot)$ *and* $h(\cdot,\cdot)$ *defined by*

$$g(\delta, y) = \frac{2}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho(\delta)](y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}Q(dx)ds \tag{3.42}$$

$$h(\delta, y) = e^{-[-2\mu/\sigma^2+\rho(\delta)]y} + \frac{2}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho(\delta)](y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}\omega(x)dxds \tag{3.43}$$

Hence $\phi_w(\delta, b)$ *is given by the expansion formula*

$$\phi_w(\delta, b) = \sum_{k=0}^\infty g^{\star k}(\delta, .) \star h(\delta, .)(\delta, b). \tag{3.44}$$

The downside is that Formula (3.44) is in practise not easy to use numerically since it involves multiple integrals due to the convolutions(as well as an infinite series which in practise is truncated). However, this approach might be more profitable than the one in Theorem 20 in cases where the scale function does not have a close form, as indeed in Theorem 20 one has to get an approximation of $W^{(\delta)}$ by inverse Laplace transform, then that of its derivative, then plug this approximation in (3.40).

Incidentally, combining both approaches theoretically gives yet another expression for the scale function. Indeed, we recall that, from Expression (4) p.19 of [KP05],

$$\mathbb{E}[e^{-\delta T_b}] = 1 + \delta\int_0^b W^{(\delta)}(y)dy - \frac{\delta}{\rho(\delta)}W^{(\delta)}(b). \tag{3.45}$$

Since the lefthandisde of (3.45) is no less that $\phi_w(\delta, b)$ with $w \equiv 1$, it has from Proposition 12 some expression of the form (3.44) expressed as a series of convoluted functions. By deriving (3.45) one observes that $W^{(\delta)}$ satisfies the first order differential equation of the form $W^{(\delta)\prime}(x) - \rho(\delta)W^{(\delta)}(x) = H(\delta, x)$ where $H(\delta, x)$ is expressed as an infinite series. The fact that $\sigma^2 > 0$ entails by Lemma 8.6 p.222 of [Kyp06] says that $W^{(\delta)}(0) = 0$, so that solving that differential equation yields the following expression for $W^{(\delta)}(x)$

$$W^{(\delta)}(x) = \int_0^x e^{-\rho(\delta)(x-y)} H(\delta, y) dy. \quad (3.46)$$

Again, Formula (3.46) is not a miracle since it involves again series with mutliple integrals. However this is to be compared with directly inverting (3.37) using a Bronwich integral, see Section 5 of [KKR13] for tricks enabling the inversion to be numerically more efficient.

3.4.2 Last passage time, and an application.

We now turn to L_b and L_b^* and focus on determining (3.33), (3.34) and (3.35). Surprisingly, there seems to be few results concerning L_b^* in the literature. As to L_b, we may mention [CY05] where the authors determine last exit times distribution, and [Bau09] where the author determine distribution of the last exit times before an exponentially distributed time. Both papers consider the class of spectrally negative Lévy processes, on the other hand the distribution is expressed in terms of Laplace transform, not of e.g. cumulative distribution function. This time, the only available option seems to be using scale functions. We have the following result.

Theorem 21. *For all $t \geq 0$ and $a \in \mathbb{R}$, $\delta \geq 0$, $b > y \geq 0$, $w > 0$, (3.33) and (3.34) have the expressions*

$$\begin{aligned}
\mathbb{P}(L_b < t) &= \int_b^\infty \mathbb{E}[D_1] W(a-b) f_{D_t}(a) da \\
\mathbb{E}[e^{-\delta L_b} 1_{\{b - D_{L_b-} \in dy,\ D_{L_b} - b \in dw\}}] &= \left[e^{\rho(\delta)(b-y)} \frac{1}{\varphi_D'(\rho(\delta))} - W^{(\delta)}(b-y) \right] dy \\
&\quad .[1 - e^{-\rho(0)w}] Q(dw + y)
\end{aligned}$$

where $f_{D_t}(.)$ is density of r.v. D_t and $W(.) = W^{(\delta)}(.)$ with $\delta = 0$. The Laplace transform (3.35) of L_b^* is given by

$$\mathbb{E}\left[e^{-\delta L_b^*}\right] = \mathbb{E}[D_1] \int_b^\infty W'(a-b) \phi(\delta, a) da$$

where $\phi(\delta, a) := \mathbb{E}[e^{-\delta T_a}] = \phi_w(\delta, a)$ with $w \equiv 1$.

In fact, a side product of Theorem 21 is that it is possible to get $\mathbb{P}(L_b \geq t,\ D_t \in da)$ for all $a \geq b$ This is a bit more general than $\mathbb{P}(L_b < t)$, and will be useful for the upcoming application. A key result and starting point that was used for establishing (3.34) is a Corollary 2 of [KPR10], which expresses the joint distribution of many quantities (that involve the last passage time, but also the minimum of D_t, $t \geq 0$, its minimum after t etc.) but that require some measures on $(0, +\infty)^2$ which are again charactarized through their (double) Laplace transform, whereas (3.34) only features scale functions. Let us also remind that in $\mathbb{E}\left[e^{-\delta L_b^*}\right]$, $\phi(\delta, a)$ is obtained as (3.45) (a function of scale function $W^{(\delta)}(.)$), or as (3.44) with $w \equiv 1$ (for the renewal approach).

We finish this section by seeing how Theorem 21 can be applied to compute degradation measures in a reliability context. We consider a component of which degradation is represented by a process $\{X_t,\ t \geq 0\}$. Lifetime is distributed as L_b without maintenance ; i.e., without maintenance, $\{X_t,\ t \geq 0\}$ has same distribution as $\{D_t,\ t \geq 0\}$, and deterioration corresponds to its last passage time. We now suppose that inspections occur at times $(U_i)_{i=1,2,...}$ such that inter inspection time verifies $U_{i+1} - U_i = m(X_{U_i+})$, where $m(.)$ is some non increasing (general) function. We also suppose that component undergoes some maintenance upon inspection if it did not fail since last inspection through some function $d : \mathbb{R} \longrightarrow \mathbb{R}$ which is some "maintenance function". On inspection at time U_i, one of the following actions is undertaken :

- either the system did not fail in interval $(U_{i-1}, U_i]$, in which case preventive maintenance occurs and degradation process evolves like $\{D_t,\ t \geq 0\}$ with initial condition $D_0 = d(x)$ up until time U_{i+1}, where x is degradation state at instant U_i- ; in other words one has $X_{U_i} = d(X_{U_i-})$,
- or the system failed in interval $(U_{i-1}, U_i]$ in which case it is repaired and degradation process starts anew, i.e. evolves like $\{D_t,\ t \geq 0\}$ with initial condition $D_0 = 0$.

What is studied is the joint distribution of the following quantities :

- the r.v. I as the first inspection after which system is completely repaired, i.e.

$$I = \inf\{i \in \mathbb{N} | \text{ failure occurred in } (U_{i-1}, U_i]\},$$

- the unavailability period of time during which component is down until next scheduled inspection :

$$\Delta^* := T^* - H_b \in [0, U_I - U_{I-1}]$$

  where $H_b \in [U_{I-1}, U_I]$ is the failure time of the component.

See Figure 3.2 for an illustration of these quantities. Their distributions (point mass probability for I or cdf for Δ^*) are available thanks to Theorem 21. Note that those could not have been

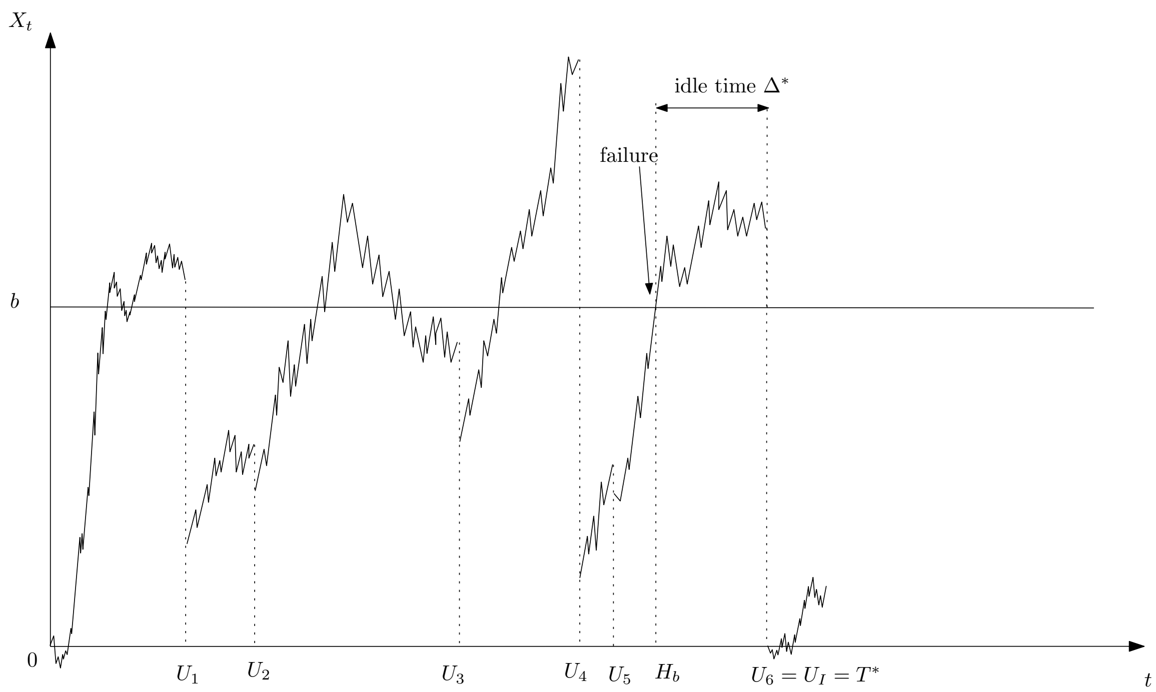

FIGURE 3.2 – Sample path of degradation process $\{X_t,\ t \geq 0\}$, with failure in $(U_5, U_6]$.

obtained if the Laplace transform of L_b had been derived instead of its cumulative distribution function (3.33).

Perspectives, Ouvertures

As a conclusion, I am presenting here some of the topics I would like to investigate (or that I am starting to investigate), some of which already involve some of my colleagues.

Multidimensional risk processes. An objective is to carry on and try find some correspondance between N dimensional risk theory and stochastic networks. A plan would first to find an adequate N dimensional equivalent of Theorem 1. This is already difficult, and a first attempt was done in the present document at the end of Section 1.3.1, however, we are that section in the presence of bounds, not equalities. Then, it would be nice to see if such a correspondance could be profitable to both sides, the "queueing theory" community, or the "ruin theory" community. For example, there exists results concerning the asymptotic distribution of a multidimensional brownian motion reflected in the orthant with a certain reflection matrix (with certain condition ensuring that this limit does exist). There exists some cases where this distribution admits a so-called "product form" which is very convenient because it is simple and interesting numerically, albeit restricted to a limited number of cases of such reflected brownian motions (see [KW95]). Since such a reflected process is a model for particular networks, this might in turn provide some information on a corresponding adequately defined multidimensional ruin problem. Conversely, some multidimensional ruin problem that were solved might give additional insight or give information on relative stochastic networks.

Reliability. This subject is related to ANR project AMMSI (`www-ljk.imag.fr/AMMSI/index.html`) in which I take part and which started in March 2012. Problem investigated in Section 3.4 is very close to what is encountered in risk theory, except maybe the application in Section 3.4.2 where notion of preventive maintenance is important. It is in fact that very notion which is specific in that field, and which is interesting to explore. There are many kinds of defining maintenance : one is to modify the state of the degradation process on inspection times (this is what happens in Section 3.4.2). Some are more subtle : for example maintenance can consist in, on inspection time, resetting the degradation process with a value which has the same value as the one at a random (properly defined) time since last inspection ; an extensive description of this ageings and maintenances can be found in [GD11]. Moreover, these maintenances may be performed differently whether the degradation process lies between two different boundaries. An example of such mechanisms may be found in [MC13].
Note that, similarly to multidimensional risk theory, there is some interest in multivariate models in reliability where failure time is defined by the entering into certain subsets of a $\mathbb{R}^N$ valued degradation process, see e.g. [MP12]. These degradation processes have, componentwise, non decreasing sample paths ; however one may imagine a situation where maintenances are performed

periodically, which leads to a model with degradation processes with negative jumps, much closer to the ones present in the present document.

Genome and branching random walks. This topic is almost completely disconnected from the previous ones and concerns modelling of evolution of transposable elements on a DNA strand. It is related to the "Projet Région" entitled "Modélisation Mathématiques des Éléments Transposables" which officially started in July 2013 and in which I participate. This is still work in progress. A transposable element is a small sequence of nucleobases, of which size is considered negligible, compared to the size of the strand. At each generation, a transposable element is either deleted, stays in place, or generates an identical twin at some random distance of which distribution varies with time. This is modelled by the following branching random walk. Let $(Z_n)_{n\in\mathbb{N}}$ be a classical Galton Watson process with offspring number having the distribution of the generic r.v. ξ with values in $\{0,1,2\}$ and distribution

$$p_i = \mathbb{P}(\xi = i), \quad i = 0,1,2.$$

We let $m := p_1 + 2p_2$ the mean number of offspring of each element, and we suppose that we are in the superciritcal case $m > 1$. Positions of the transposable elements at generation n are denoted by X_k^n, $k = 1, ..., Z_n$, on the event of non extinction, which take their values in some set $\mathbb{S}$. We let V_k^n the distance between elements k at time n and its future offspring (if any), and we suppose that for each n, sequence $(V_k^n)_{k\in\mathbb{N}}$ is identically distributed (not necessarily independent) as a random variable V^n that varies with n. In this work we are interested in the two following empirical measures and their random generating functions

$$\begin{aligned} \nu_n &:= \frac{1}{Z_n} \mathbb{1}_{\{Z_n>0\}} \sum_{k=1}^{Z_n} \delta_{X_k^n}, \quad M_n(u) := \frac{1}{Z_n} \mathbb{1}_{\{Z_n>0\}} \sum_{k=1}^{Z_n} e^{uX_k^n}, \; u \in \mathbb{R}, \\ \hat{\nu}_n &:= \frac{1}{m^n} \mathbb{1}_{\{Z_n>0\}} \sum_{k=1}^{Z_n} \delta_{X_k^n}, \quad \hat{M}_n(u) := \frac{1}{m^n} \mathbb{1}_{\{Z_n>0\}} \sum_{k=1}^{Z_n} e^{uX_k^n}, \; u \in \mathbb{R}, \end{aligned} \tag{3.47}$$

and their almost sure limit ν_∞ and $\hat{\nu}_\infty$ as $n \to +\infty$, which is equivalent to the pointwise convergence of $M_n(u)$ and $\hat{M}_n(u)$. When the V^n's have identical distribution with moment generating function $\varphi(u) = \mathbb{E}(e^{uV^n})$, $\hat{M}_n(u)$ is related to *Biggin's martingale* $W_n(u) = \frac{1}{m(u)^n}\mathbb{1}_{\{Z_n>0\}} \sum_{k=1}^{Z_n} e^{uX_k^n}$ where $m(u)$ is a renormalizing factor which here has the simple expression $m(u) := p_1 + p_2\varphi(u)$. So far in this project we have done or are doing the following :

1. We have exhibited a martingale related to $M_n(u)$ which is different from Biggin's martingale. By standard positive martingale and bounded submartingale theory, this entails the convergence of $M_n(u)$ as $n \to +\infty$, when the V^n decrease fast enough as n becomes large, to some $M_\infty(u)$.
2. We are trying to give some characteristic on $M_\infty(u)$. For the moment, we are trying to find how to obtain some expression of its expectation $\mathbb{E}(M_\infty(u))$ by the standard method of conditioning with respect to the state of the branching random walk at generation $n = 1$.

Two kinds of sets $\mathbb{S}$ have been studied in the above points : $\mathbb{S} = \mathbb{R}$, and $\mathbb{S} = \mathbb{T}$, the torus on $[0,1]$. In the case $\mathbb{S} = \mathbb{T}$, distribution ov the V^n's may be assumed constant ; in fact one practical aspect when it is exponentially or Laplace distributed, as shown on Figure (3.3). Let us note that one central tool for studying Point 2) is the so-called *Many to one Lemma* (see Lemma 2.1 of [Mal13]) which links distribution branching random walks to one dimensional classical random walks.

Furthermore, we also study the model where particles are potentially trapped in some subsets of $\mathbb{S}$, i.e. such that, whenever an element is spawned in one of these subsets then all its descendants are located in its exact place. This is motivated by the fact that, in practice, a transposable element may be duplicated in a "dead" zone on the DNA strand, which is not likely to produce such elements in the future. Using the Many to one Lemma, we are trying to find an integral equation satisfied by x $\mathbb{E}(M_\infty^x(u))$ (where x is position of the initial element) when $\mathbb{S} = \mathbb{T}$ and/or when there is such a dead zone.

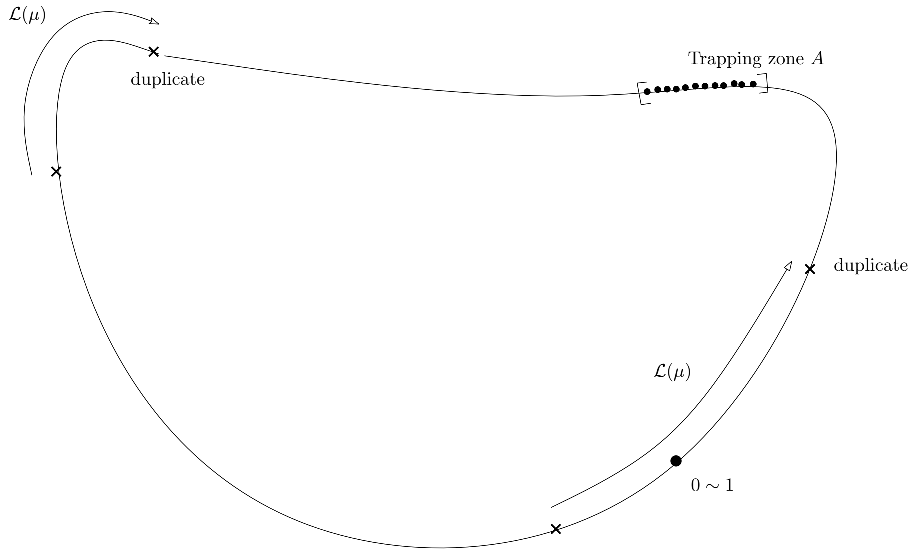

FIGURE 3.3 – Duplication mechanism in the torus case $\mathbb{S} = \mathbb{T}$, when displacements are Laplace distributed.

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